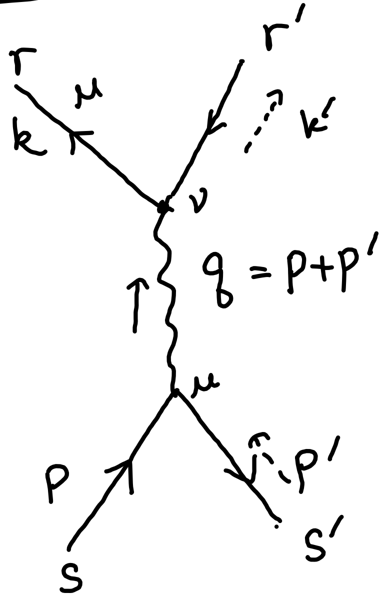
Chap 5. Quantum Electrodynamics (QED)

5.1. $e^+e^- \to \mu^+\mu^-$

$$\langle \mu^-(k)\,\mu^+(k')|\ (-ie)^2\,(\bar\psi_\mu \gamma^\mu \psi_\mu A_\mu)_x\,(\bar\psi_e \gamma^\nu \psi_e A_\nu)_y\,|e^-(p)\,e^+(p')\rangle$$

$i\mathcal{M} =$



$$= \bar v^{s'}(p')\,(-ie\gamma^\mu)\,u^s(p)\,\frac{-ig_{\mu\nu}}{q^2}$$

e^-

$$\times\ \bar u^r(k)\,(-ie\gamma^\nu)\,v^{r'}(k')$$

μ^-

$$= \frac{ie^2}{q^2}\,\underbrace{\left[\bar v^{s'}(p')\,\gamma^\mu\,u^s(p)\right]_e}_{\text{scalar}}\,\underbrace{\left[\bar u^r(k)\,\gamma_\mu\,v^{r'}(k')\right]_\mu}_{\text{scalar}}$$

$$\underset{\sim\sim}{|\mathcal{M}|^2} = \frac{i e^2}{q^2} \left[\bar{v}^{s'}(p') \gamma^{\underline{\underline{\mu}}} u^s(p)\right]_e \left[\bar{u}^r(k) \gamma_{\underline{\underline{\mu}}} v^{r'}(k')\right]_\mu$$

$$\times \frac{(-i) e^2}{q^2} \underbrace{\left[\bar{v}^{s'}(p') \gamma^\nu u^s(p)\right]_e^\dagger}_{u^{s\dagger} \gamma^{\nu\dagger} \gamma^{0\dagger} v^{s'}} \underbrace{\left[\bar{u}^r(k) \gamma^\nu v^{r'}(k')\right]_\mu^\dagger}_{= \bar{v}^{r'}(k') \gamma_\nu u^r(k)}$$

$$u^{s\dagger} \to \gamma^0\gamma^0$$

$$= \bar{u}^s \underbrace{\gamma^0 \gamma^{\nu\dagger} \gamma^0}_{} v^{s'}$$

$$\left.\begin{matrix} \mu = 0 & \gamma^0 \\ \mu \neq 0 & \gamma^0(-\vec{\gamma})\gamma^0 = \vec{\gamma} \end{matrix}\right\} \gamma^\nu$$

$$= \bar{u}^s(p) \gamma^\nu v^{s'}(p')$$

$$= \frac{e^4}{q^4} \left[\bar{v}^{s'}(p') \gamma^\mu u^s(p)\right]_e \left[\bar{u}^s(p) \gamma^\nu v^{s'}(p')\right]_e$$

$$\left[\bar{u}^r(k) \gamma_\mu v^{r'}(k')\right]_\mu \left[\bar{v}^{r'}(k') \gamma_\nu u^r(k)\right]_\mu$$

Spin sum of "unpolarized beams"

~~pure~~ : $\left| \frac{1}{\sqrt{2}} |\uparrow_{e^-}\rangle + \frac{1}{\sqrt{2}} |\downarrow_{e^-}\rangle \right|^2$

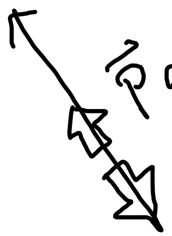
*mixed ; $\frac{1}{2} \left| |\uparrow e_-\rangle \right|^2 + \frac{1}{2} \left| |\downarrow e_-\rangle \right|^2$

$\downarrow \quad \frac{1}{2} \sum_{s=\uparrow,\downarrow} \left| |e_-^s\rangle \right|^2$

$(\quad)(\quad)\boxed{()(\quad)}\cdots$

$$= \left(\frac{1}{2}\right)^4 \sum_{s} \sum_{s'} \sum_{r} \sum_{r'} \frac{e^4}{q^4} \left[\bar{v}^{s'}(p') \left[\gamma^\mu \underbrace{u^s(p)\Big]_e \Big[\bar{u}^s(p)}_{(\not{p}+m_e)} \gamma^\nu\right]_A v^{s'}(p')\right]_e$$

$$\left[\bar{u}^r(k)\, \gamma_\mu\, \underbrace{v^{r'}(k')\Big]_\mu \Big[\bar{v}^{r'}(k')}_{\not{k}'-m_\mu}\, \gamma_\nu\, u^r(k)\right]_\mu$$



$\vec{p} \propto \hat{n}$

$\vec{\sigma}\cdot\hat{n} = h = 1 \ (R)$

$\quad\quad\quad\quad\quad -1 \ (L)$

$$\sum_{s=\uparrow,\downarrow} u^s(p)\bar{u}^s(p) = \not{p} + m$$

$$\sum_{s=\uparrow,\downarrow} v^s(p)\,\bar{v}^s(p) = \not{p} - m$$

$$\vec{u}^T \cdot A \cdot \vec{v} = \begin{pmatrix} & \vec{u}^T & \end{pmatrix} \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} \\ \vec{v} \\ \\ \end{pmatrix} = \sum_{i,j} u_i A_{ij} v_j$$

$$\overset{?}{\|}$$

$$\sum_{i,k} \delta_{ik} \left[\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \underbrace{\begin{pmatrix} \\ \vec{v} \\ \\ \end{pmatrix} \begin{pmatrix} & \vec{u}^T & \end{pmatrix}}_{B_{jk} = v_j u_k} \right]_{ik} = \sum_j A_{ij} B_{jk} = \sum_j A_{ij} v_j u_k \times \boxed{\times \sum_{i,k} \delta_{ik}}$$

$$\| $$

$$\sum_{i,k} \delta_{ik} (AB)_{ik} = \sum_i (AB)_{ii} = tr(AB)$$

$$\therefore \boxed{\vec{u}^T A \vec{v} = tr\big(A \overbrace{(\vec{v}\ \vec{u}^T)}^{B}\big)}$$

$$\left(\frac{1}{2}\right)^4 \sum_{s'} \sum_{r} \frac{e^4}{q^4} \Big[\bar v^{s'}(p') \Big[\gamma^\mu \underbrace{u^s(p)\Big]_e \Big[\bar u^s(p)}_{(\not p + m_e)} \gamma^\nu \Big]_A v^{s'}(p') \Big]_e$$

$$\Big[\bar u^r(k) \Big[\gamma_\mu \underbrace{v^{r'}(k)\Big]_\mu \Big[\bar v^{r'}(k')}_{} \gamma_\nu \Big] u^r(k) \Big]_\mu$$

$$= \left(\frac{1}{2}\right)^4 \frac{e^4}{q^4} \operatorname{tr}\Big[\Big[\gamma^\mu \underbrace{u^s(p)\Big]_e \Big[\bar u^s(p)}_{(\not p + m_e)} \gamma^\nu \Big]_A \underbrace{\sum_{s'} v^{s'}(p') \bar v^{s'}(p')}_{\not p' - m_e} \Big]$$

$$\times \operatorname{tr}\Big[\Big[\gamma_\mu \underbrace{v^{r'}(k)\Big]_\mu \Big[\bar v^{r'}(k')}_{\not k' - m_\mu} \gamma_\nu \Big] \underbrace{\sum_{r} u^r(k) \bar u^r(k)}_{\not k + m_\mu} \Big]$$

$$= \frac{e^4}{16} \frac{1}{q^4} \operatorname{tr}\left(\gamma^\mu (\not p + m_e) \gamma^\nu (\not p' - m_e) \right)$$

$$\times \operatorname{tr}\left(\gamma_\mu (\not k' - m_\mu) \gamma_\nu (\not k + m_\mu) \right)$$

Trace technology (spin-sum formulas)

- $\mathrm{tr}(\text{odd } \gamma\text{'s}) = 0$

$$\gamma^{5\,2} = \mathbb{1}$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

$$\mathrm{tr}(\gamma^5\gamma^5 \overbrace{\gamma\,\gamma \cdots \gamma}^{\text{odd } (n \geq 1)}) = (-1)^n\, \mathrm{tr}(\underbrace{\gamma^5 \gamma \cdots \gamma}_{A}\, \underbrace{\gamma^5}_{B})$$

$(-1)^n \gamma^5$

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

$$= \underbrace{(-1)^n}_{-1}\, \mathrm{tr}(\underbrace{\gamma^5 \gamma^5}_{\mathbb{1}} \gamma \cdots \gamma)$$

- $\mathrm{tr}(\mathbb{1}) = 4$

- $$\underbrace{\mathrm{tr}(\gamma^\mu \gamma^\nu)}_{= g^{\mu\nu}\cdot 4} + \mathrm{tr}(\gamma^\nu \gamma^\mu) = \mathrm{tr}(\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{= 2g^{\mu\nu}\mathbb{1}}) = 2 g^{\mu\nu}\cdot 4$$

$$\underline{\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma)} = 2 g^{\mu\nu}\, \mathrm{tr}(\gamma^\rho \gamma^\sigma) + (-)\, \mathrm{tr}(\gamma^\nu \gamma^\mu \gamma^\rho \gamma^\sigma)$$

$$= -2\, g^{\mu\rho}\, \mathrm{tr}(\gamma^\nu \gamma^\sigma) + (-)^2\, \mathrm{tr}(\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma)$$

$$+ 2\, g^{\mu\sigma}\, \mathrm{tr}(\gamma^\nu \gamma^\rho) + (-1)^3\, \mathrm{tr}(\underbrace{\gamma^\nu \gamma^\rho \gamma^\sigma}_{A} \underbrace{\gamma^\mu}_{B})$$

$$-1 \quad \mathrm{tr}(BA)$$

$$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu + 2 g^{\mu\nu} \mathbb{1}$$

$$= 4 \cdot (g^{\mu\nu} \cdot g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$$

$$\gamma^\mu \gamma_\mu = 4\, \mathbb{1}$$

$$\underbrace{\gamma^\mu \gamma^\nu}_{2 g^{\mu\nu} \mathbb{1} - \gamma^\nu \gamma^\mu} \gamma_\mu = -2\, \gamma^\nu \qquad \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4 g^{\nu\rho}$$

$$\|$$

$$\underbrace{2 g^{\mu\nu} \gamma_\mu}_{\gamma^\nu} - \gamma^\nu \underbrace{\gamma^\mu \gamma_\mu}_{4\, \mathbb{1}} = -2\, \gamma^\nu$$

$$\mathrm{tr}\Big(\gamma^{\mu} (\underbrace{\not{p}}_{= p_\alpha \gamma^\alpha} + m_e)\, \gamma^{\nu} (\underbrace{\not{p}'}_{= p'_\beta \gamma^\beta} - m_e) \Big)$$

$$= - m_e^2\, 4 g^{\mu\nu} + p_\alpha p'_\beta \underbrace{\mathrm{tr}(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta)}_{4 (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\alpha\nu})}$$

$$4 (p^\mu p'^\nu + p^\nu p'^\mu)$$

$$-4 g^{\mu\nu} (p \cdot p' + m_e^2)$$

$$= 4 (p^\mu p'^\nu + p^\nu p'^\mu) - 4 g^{\mu\nu} (p \cdot p' + m_e^2)$$

$$\mathrm{tr}\Big(\gamma_\mu (\not{k}' - m_\mu) \gamma_\nu (\not{k} + m_\mu) \Big)$$

$$= 4 (k'_\mu k_\nu + k'_\nu k_\mu) - 4 g_{\mu\nu} (k \cdot k' + m_\mu^2)$$

$$\left\{ \underbrace{4\left(p^\mu p'^\nu + p^\nu p'^\mu\right)} - 4 g^{\mu\nu}\left(p\cdot p' + m_e^2\right)\right\}$$

$$\times \left\{ \underbrace{4\left(k'_\mu k_\nu + k'_\nu k_\mu\right)} - 4 g_{\mu\nu}\underbrace{\left(k\cdot k' + m_\mu^2\right)}\right\}$$

$m_e = 0.5$ MeV $\qquad m_e^2 \ll m_\mu^2 \quad \to \quad m_e^2 \approx 0$

$m_\mu = 120$ MeV

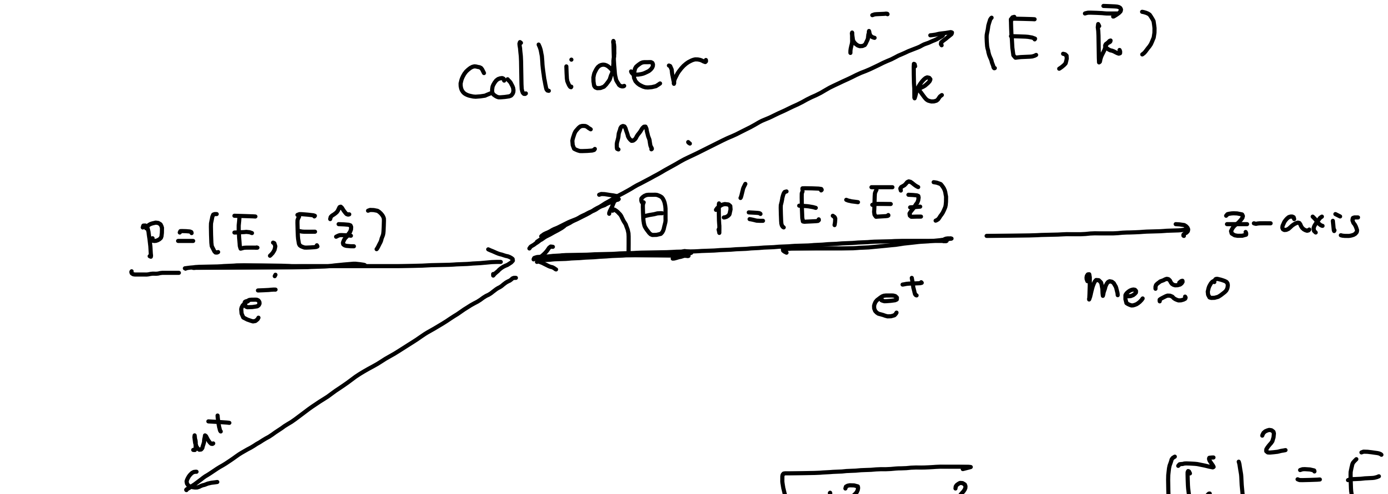
$$= 16\left\{ 2(p\cdot k')(p'\cdot k) + 2(p\cdot k)(p'\cdot k') - \underbrace{2(p\cdot p')(\cancel{k\cdot k'} + m_\mu^2)} - \underbrace{2(k\cdot k')}\cancel{(p\cdot p')} + \underbrace{\underbrace{4(p\cdot p')(\cancel{k\cdot k'} + m_\mu^2)}}_{2 m_\mu^2 (p\cdot p')} \right\}$$

$$= 32\left((p\cdot k')(p'\cdot k) + (p\cdot k)(p'\cdot k') + m_\mu^2 (p\cdot p')\right)$$

$$\times \frac{1}{4}\frac{e^4}{q^4}$$

$\left(\frac{1}{2}\right)\sum_s^*$

↑ final state; not average but sum



$k' = (E, -\vec{k})$

$$E = \sqrt{\vec{k}^2 + m_\mu^2} \rightarrow (|\vec{k}|)^2 = E^2 - m_\mu^2$$

$$p \cdot p' = E^2 + E^2 = 2E^2$$

$$q^2 = (p+p')^2 = \underset{m_e^2 \approx 0}{\underset{\parallel}{p^2}} + \underset{m_e^2 \approx 0}{\underset{\parallel}{p'^2}} + 2p \cdot p' \approx 2p \cdot p' = 4E^2$$

$$p \cdot k = E^2 - E|\vec{k}|\cos\theta = p' \cdot k'$$

$$p \cdot k' = E^2 + E|\vec{k}|\cos\theta = p' \cdot k$$

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{4E^2}\left(\underbrace{\left(E^2 + E|\vec{k}|\cos\theta\right)^2 + \left(E^2 - E\,|\vec{k}|\cos\theta\right)^2}_{} + m_\mu^2\, 2E^2\right)$$

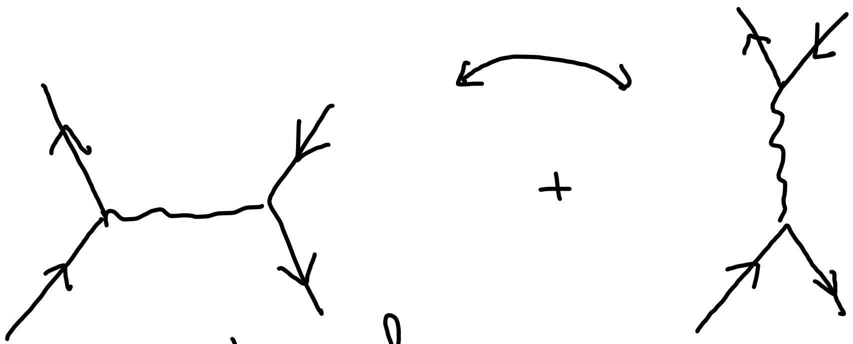
$$2E^4 + 2E^2\underbrace{|\vec{k}|^2}_{E^2 - m_\mu^2}\cos^2\theta + 2E^2 m_\mu^2 \quad (E^2 - m_\mu^2)\cos^2\theta$$

$$= 2E^2\left((E^2 + m_\mu^2)\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{cm}^2} \frac{|\vec{k}|}{16\pi^2 E_{cm}} \overline{|\mathcal{M}|^2}$$

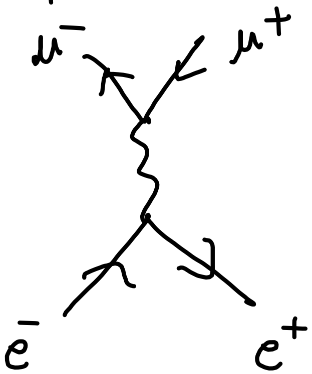
$$E_{cm} = 2E$$

$e^+e^- \to e^+e^-$



s-channel　　　　　　　　t-channel

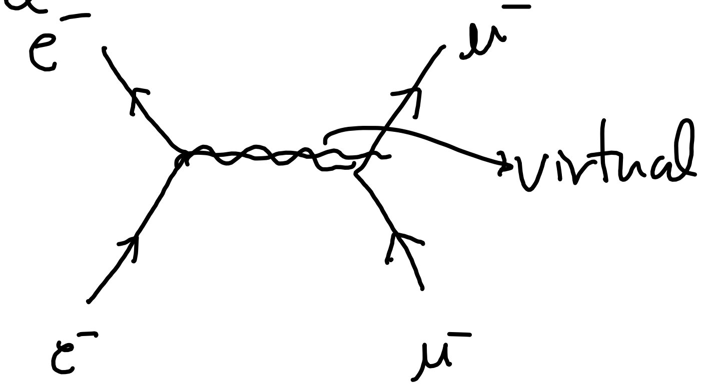
$e^+e^- \to \mu^+\mu^-$

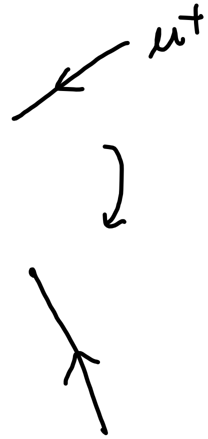


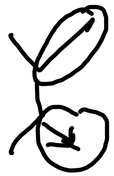
$s \leftrightarrow t$ channel

"Crossing symmetry"

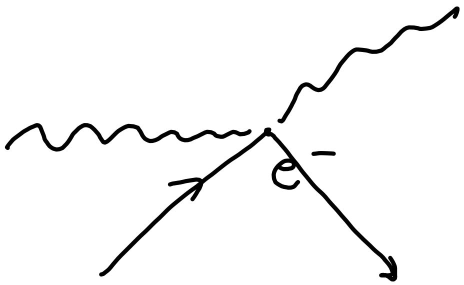
Cross chanel

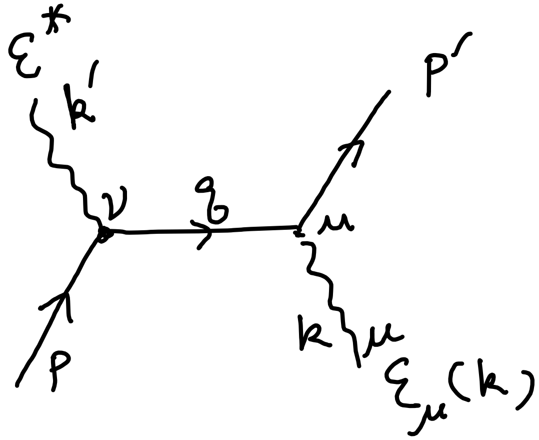






Compton scattering: phonton external lines





$$\epsilon^*_\nu(k')\, \bar{u}(p')\, \gamma^\mu \frac{i(\not{q}+m)}{q^2-m^2} \gamma^\nu u(p)\, \epsilon_\mu(k)$$