

Finite-size effects for single spike

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ABSTRACT: We use the reduction of the string dynamics on $R_t \times S^3$ to the Neumann-Rosochatius integrable system to map *all* string solutions described by this dynamical system onto solutions of the complex sine-Gordon integrable model. This mapping relates the parameters in the solutions on both sides of the correspondence. In the framework of this approach, we find finite-size string solutions, their images in the (complex) sine-Gordon system, and the leading finite-size effects of the single spike “ $E - \Delta\varphi$ ” relation for both $R_t \times S^2$ and $R_t \times S^3$ cases.

KEYWORDS: AdS-CFT Correspondence, Integrable Equations in Physics.

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1. Introduction

Recent developments in AdS/CFT correspondence between type IIB strings on $AdS_5 \times S^5$ and its dual $\mathcal{N} = 4$ super Yang-Mills (SYM) theories [1] are mainly based on the integrabilities discovered in both theories. Integrability of the SYM side appears in the calculations of conformal dimensions which are related to the string energies according to the AdS/CFT correspondence. A remarkable observation by Minahan and Zarembo [2] is that the conformal dimension of an operator composed of scalar fields in the $\mathcal{N} = 4$ SYM in the planar limit can be computed by diagonalizing the Hamiltonian of one-dimensional integrable spin chain model. This task can be done by solving a set of coupled algebraic equations called Bethe ansatz equations. It has been shown that explicit calculation of the eigenvalues for various SYM operators agree with those computed from the SYM perturbation theory.

This result has been further extended to the full $PSU(2,2|4)$ sector [3] and the Bethe ansatz equations which are supposed to hold for all loops are conjectured [4, 5].

The string side of the correspondence is mostly studied at the classical level due to the lack of full quantization. The type IIB string theory on $AdS_5 \times S^5$ is described by a nonlinear sigma model with $PSU(2,2|4)$ symmetry [6]. This sigma model has been shown to have an infinite number of local and nonlocal conserved currents [7] and some of the conserved charges such as energy and angular momentum are computed explicitly from the classical integrability (see, for example, [8] and the references therein). These results based on the classical integrability provide valuable information on the AdS/CFT duality in the domain of large t'Hooft coupling constant. A new direction to quantizing the string theory is to find exact S -matrix between the fundamental spectrum of the theory on the world sheet. It has been shown that the S -matrix along with the exact particle spectrum can be determined by the underlying symmetry and integrability in the theory [9, 10]. The overall scalar factor of the S -matrix, sometimes called as dressing phase, which can not be determined by the symmetry alone, has been computed exactly [11, 12] from the crossing relation [13]. In this process, the explicit expression of the dressing phase in the classical limit, which was determined by the classical integrability, was essential [14].

Various classical solutions play an important role in testing and understanding the correspondence. The classical giant magnon (GM) state [15] discovered in $R_t \times S^2$ gives a strong support for the conjectured all-loop $SU(2)$ spin chain and makes it possible to get a deep insight in the AdS/CFT duality. In addition, this solution is related to classical sine-Gordon (SG) model which provides a geometric understanding of the string in curved space. This is extended to the magnon bound state which corresponds to a string moving on $R_t \times S^3$ and related to the complex sine-Gordon (CSG) model [16, 17]. Further extensions to $R_t \times S^5$ have been also worked out [18–21]. Another interesting classical string solution is the spiky string which has been first found in the AdS space [22] and in the S^5 [23]. A particular case of these states is the single spike (SS) which describes a string which is wrapping infinitely around the equator with a spike in the middle. This has been investigated in a static gauge using the Nambu-Goto action in S^2 and S^3 by Ishizeki and Kruczenski [24]. In addition, they have shown that both the GM and the SS solutions on S^2 can be related to the classical SG equation. This is possible by reformulating the problem in a conformal gauge using the Polyakov action and assuming a particular ansatz for string coordinates which leads to the well-known one-dimensional integrable Neumann-Rosochatius (NR) system. This approach has been previously developed and applied to find classical solutions in [25, 26, 18]. The application of the NR system to the SS in S^3 has been worked out in [27]. Also a semiclassical quantization of the SS has been recently studied in [28]

More recently the finite-size correction, or the Lüscher correction, is actively investigated as a new window for the AdS/CFT correspondence. The integrable spin chains based on the Bethe ansatz are showing several limitations such as the wrapping problem which occurs in dealing with a composite operator of a finite length in a strong t'Hooft coupling limit. It is quite important to compute the conformal dimensions of such operators and compare with the energies of the string states with all other quantum numbers finite. One

way of confirming the S -matrix is to derive the Lüscher correction from the S -matrix and compare with the classical string result. The finite-size effects for the GM have been computed from the S -matrix [29] and are shown to be consistent with classical string results. Finite-size effects for the GM has been first found by solving the string sigma model in a uniform and conformal gauges [30] and, subsequently, many related results, such as gauge independence [31], multi GM states [32] and quantization of finite-size GM [33], have been derived.¹ This result has been also related to explicit solutions of the SG equation in a finite-size space [34].

In this article we compute the finite-size effects for the SS in both S^2 and S^3 . The finite-size SS solutions were related to the helical string solutions without analyzing the corrections to the energy-charge dispersion relations in [35]. While there are some conjectures for the quantum SS from the integrable spin chain models such as the Hubbard model [24] or antiferromagnetic $SO(6)$ spin chain model [35], the S -matrix for the SS is known only in the classical limit [36] and still not available at the full quantum level. This excludes the S -matrix approach for the Lüscher correction and leaves the classical analysis as a viable option.

The paper is organized as follows. In section 2 we introduce the classical string action on $R_t \times R^3$ and the corresponding NR system. This system is shown to be equivalent to a particular case of the CSG equation in section 3. In section 4 we provide our main result on the finite-size effects of the SS on both $R_t \times S^2$ and $R_t \times S^3$. We conclude the paper with some remarks in section 5 and with appendices containing the explicit relationship between the NR system and CSG for the GM and SS in infinite-size system and ϵ -expansions for elliptic functions and relevant coefficients.

2. Strings on $R_t \times S^3$ and the NR integrable system

Let us start with the Polyakov string action

$$S^P = -\frac{T}{2} \int d^2\xi \sqrt{-\gamma} \gamma^{ab} G_{ab}, \quad G_{ab} = g_{MN} \partial_a X^M \partial_b X^N, \quad (2.1)$$

$$\partial_a = \partial / \partial \xi^a, \quad a, b = (0, 1), \quad (\xi^0, \xi^1) = (\tau, \sigma), \quad M, N = (0, 1, \dots, 9),$$

and choose *conformal gauge* $\gamma^{ab} = \eta^{ab} = \text{diag}(-1, 1)$, in which the Lagrangian and the Virasoro constraints take the form

$$\mathcal{L}_s = \frac{T}{2} (G_{00} - G_{11}) \quad (2.2)$$

$$G_{00} + G_{11} = 0, \quad G_{01} = 0. \quad (2.3)$$

where T is the string tension.

We embed the string in $R_t \times S^3$ subspace of $AdS_5 \times S^5$ as follows

$$Z_0 = R e^{it(\tau, \sigma)}, \quad W_j = R r_j(\tau, \sigma) e^{i\varphi_j(\tau, \sigma)}, \quad \sum_{j=1}^2 W_j \bar{W}_j = R^2,$$

¹Some topics related to finite-size effects were also considered in [42].

where R is the common radius of AdS_5 and S^5 , and t is the AdS time. For this embedding, the metric induced on the string worldsheet is given by

$$G_{ab} = -\partial_{(a}Z_0\partial_{b)}\bar{Z}_0 + \sum_{j=1}^2 \partial_{(a}W_j\partial_{b)}\bar{W}_j = R^2 \left[-\partial_a t \partial_b t + \sum_{j=1}^2 (\partial_a r_j \partial_b r_j + r_j^2 \partial_a \varphi_j \partial_b \varphi_j) \right].$$

The corresponding string Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_s + \Lambda_s \left(\sum_{j=1}^2 r_j^2 - 1 \right),$$

where Λ_s is a Lagrange multiplier. In the case at hand, the background metric does not depend on t and φ_j . Therefore, the conserved quantities are the string energy E_s and two angular momenta J_j , given by

$$E_s = - \int d\sigma \frac{\partial \mathcal{L}_s}{\partial(\partial_0 t)}, \quad J_j = \int d\sigma \frac{\partial \mathcal{L}_s}{\partial(\partial_0 \varphi_j)}. \quad (2.4)$$

It is known that restricting ourselves to the case

$$\begin{aligned} t(\tau, \sigma) &= \kappa\tau, & r_j(\tau, \sigma) &= r_j(\xi), & \varphi_j(\tau, \sigma) &= \omega_j\tau + f_j(\xi), \\ \xi &= \alpha\sigma + \beta\tau, & \kappa, \omega_j, \alpha, \beta &= \text{constants}, \end{aligned} \quad (2.5)$$

reduces the problem to solving the NR integrable system [18]. For the case under consideration, the NR Lagrangian reads (prime is used for $d/d\xi$)

$$L_{NR} = (\alpha^2 - \beta^2) \sum_{j=1}^2 \left[r_j'^2 - \frac{1}{(\alpha^2 - \beta^2)^2} \left(\frac{C_j^2}{r_j^2} + \alpha^2 \omega_j^2 r_j^2 \right) \right] + \Lambda_s \left(\sum_{j=1}^2 r_j^2 - 1 \right), \quad (2.6)$$

where the parameters C_j are integration constants after single time integration of the equations of motion for $f_j(\xi)$:

$$f_j' = \frac{1}{\alpha^2 - \beta^2} \left(\frac{C_j}{r_j^2} + \beta\omega_j \right). \quad (2.7)$$

The constraints (2.3) give the conserved Hamiltonian H_{NR} and a relation between the embedding parameters and the arbitrary constants C_j :

$$H_{NR} = (\alpha^2 - \beta^2) \sum_{j=1}^2 \left[r_j'^2 + \frac{1}{(\alpha^2 - \beta^2)^2} \left(\frac{C_j^2}{r_j^2} + \alpha^2 \omega_j^2 r_j^2 \right) \right] = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \kappa^2, \quad (2.8)$$

$$\sum_{j=1}^2 C_j \omega_j + \beta \kappa^2 = 0. \quad (2.9)$$

For closed strings, r_j and f_j satisfy the following periodicity conditions

$$r_j(\xi + 2\pi\alpha) = r_j(\xi), \quad f_j(\xi + 2\pi\alpha) = f_j(\xi) + 2\pi n_\alpha, \quad (2.10)$$

where n_α are integer winding numbers. On the ansatz (2.5), E_s and J_j introduced in (2.4) take the form

$$E_s = \frac{\sqrt{\lambda} \kappa}{2\pi \alpha} \int d\xi, \quad J_j = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{\alpha^2 - \beta^2} \int d\xi \left(\frac{\beta}{\alpha} C_j + \alpha \omega_j r_j^2 \right), \quad (2.11)$$

where we have used that the string tension and the 't Hooft coupling constant λ are related by $TR^2 = \frac{\sqrt{\lambda}}{2\pi}$.

In order to identically satisfy the embedding condition

$$\sum_{j=1}^2 r_j^2 - 1 = 0,$$

we introduce a new variable $\theta(\xi)$ by

$$r_1(\xi) = \sin \theta(\xi), \quad r_2(\xi) = \cos \theta(\xi). \quad (2.12)$$

Then, eq. (2.8) leads to

$$\begin{aligned} \theta'(\xi) &= \pm \frac{1}{\alpha^2 - \beta^2} \left[(\alpha^2 + \beta^2) \kappa^2 - \frac{C_1^2}{\sin^2 \theta} - \frac{C_2^2}{\cos^2 \theta} - \alpha^2 (\omega_1^2 \sin^2 \theta + \omega_2^2 \cos^2 \theta) \right]^{1/2} \\ &\equiv \pm \frac{1}{\alpha^2 - \beta^2} \Theta(\theta), \end{aligned} \quad (2.13)$$

which can be integrated to give

$$\xi(\theta) = \pm (\alpha^2 - \beta^2) \int \frac{d\theta}{\Theta(\theta)}. \quad (2.14)$$

From eqs. (2.7) and (2.12), we can obtain

$$f_1 = \frac{\beta \omega_1 \xi}{\alpha^2 - \beta^2} \pm C_1 \int \frac{d\theta}{\sin^2 \theta \Theta(\theta)}, \quad (2.15)$$

$$f_2 = \frac{\beta \omega_2 \xi}{\alpha^2 - \beta^2} \pm C_2 \int \frac{d\theta}{\cos^2 \theta \Theta(\theta)}. \quad (2.16)$$

Let us also point out that the solutions for $\xi(\theta)$ and f_j must satisfy the conditions (2.9) and (2.10). All these solve formally the NR system for the present case.

As far as we are searching for real solutions, the expressions under the square roots in (2.13) must be positive, which put restrictions on the possible values of the parameters. Of course, this condition arises from the requirement that the NR Hamiltonian (2.8) should be positive.

3. Relationship between the NR and CSG systems

Due to Pohlmeyer [37], we know that the string dynamics on $R_t \times S^3$ can be described by the CSG equation. In this section, we derive the relation between the solutions of the two integrable systems.

The CSG system is defined by the Lagrangian

$$\mathcal{L}(\psi) = \frac{\eta^{ab} \partial_a \bar{\psi} \partial_b \psi}{1 - \bar{\psi} \psi} + M^2 \bar{\psi} \psi$$

which give the equation of motion

$$\partial_a \partial^a \psi + \bar{\psi} \frac{\partial_a \psi \partial^a \psi}{1 - \bar{\psi} \psi} - M^2 (1 - \bar{\psi} \psi) \psi = 0.$$

If we represent ψ in the form

$$\psi = \sin(\phi/2) \exp(i\chi/2),$$

the Lagrangian can be expressed as

$$\mathcal{L}(\phi, \chi) = \frac{1}{4} \left[\partial_a \phi \partial^a \phi + \tan^2(\phi/2) \partial_a \chi \partial^a \chi + (2M)^2 \sin^2(\phi/2) \right],$$

along with the equations of motion

$$\partial_a \partial^a \phi - \frac{1}{2} \frac{\sin(\phi/2)}{\cos^3(\phi/2)} \partial_a \chi \partial^a \chi - M^2 \sin \phi = 0, \quad (3.1)$$

$$\partial_a \partial^a \chi + \frac{2}{\sin \phi} \partial_a \phi \partial^a \chi = 0. \quad (3.2)$$

The SG system corresponds to a particular case of $\chi = 0$.

To relate the NR system with the CSG integrable system, we consider the case

$$\phi = \phi(\xi), \quad \chi = A\sigma + B\tau + \tilde{\chi}(\xi),$$

where ϕ and $\tilde{\chi}$ depend on only one variable $\xi = \alpha\sigma + \beta\tau$ in the same way as in our NR ansatz (2.5). Then the equations of motion (3.1), (3.2) reduce to

$$\phi'' - \frac{1}{2} \frac{\sin(\phi/2)}{\cos^3(\phi/2)} \left[\tilde{\chi}'^2 + 2 \frac{A\alpha - B\beta}{\alpha^2 - \beta^2} \tilde{\chi}' + \frac{A^2 - B^2}{\alpha^2 - \beta^2} \right] - \frac{M^2 \sin \phi}{\alpha^2 - \beta^2} = 0, \quad (3.3)$$

$$\tilde{\chi}'' + \frac{2\phi'}{\sin \phi} \left(\tilde{\chi}' + \frac{A\alpha - B\beta}{\alpha^2 - \beta^2} \right) = 0. \quad (3.4)$$

We further restrict ourselves to the case of $A\alpha = B\beta$. A trivial solution of eq. (3.4) is $\tilde{\chi} = \text{constant}$, which corresponds to the solutions of the CSG equations considered in [17, 38] for a GM string on $R_t \times S^3$. More nontrivial solution of (3.4) is

$$\tilde{\chi} = C_\chi \int \frac{d\xi}{\tan^2(\phi/2)}. \quad (3.5)$$

The replacement of the above into (3.3) gives

$$\phi'' = \frac{M^2 \sin \phi}{\alpha^2 - \beta^2} + \frac{1}{2} \left[C_\chi^2 \frac{\cos(\phi/2)}{\sin^3(\phi/2)} - \frac{A^2 \sin(\phi/2)}{\beta^2 \cos^3(\phi/2)} \right]. \quad (3.6)$$

Integrating once, we obtain

$$\phi' = \pm \left[\left(C_\phi - \frac{2M^2}{\alpha^2 - \beta^2} \right) + \frac{4M^2}{\alpha^2 - \beta^2} \sin^2(\phi/2) - \frac{A^2/\beta^2}{1 - \sin^2(\phi/2)} - \frac{C_\chi^2}{\sin^2(\phi/2)} \right]^{1/2} \quad (3.7)$$

$$\equiv \pm \Phi(\phi),$$

from which we get

$$\xi(\phi) = \pm \int \frac{d\phi}{\Phi(\phi)}, \quad \chi(\phi) = \frac{A}{\beta} (\beta\sigma + \alpha\tau) \pm C_\chi \int \frac{d\phi}{\tan^2(\phi/2)\Phi(\phi)}.$$

All these solve the CSG system for the considered particular case. It is clear from (3.7) that the expression inside the square root must be positive.

Now we are ready to establish a correspondence between the NR and CSG integrable systems described above. To this end, we make the following identification

$$\sin^2(\phi/2) \equiv \frac{\sqrt{-G}}{K^2} \quad (3.8)$$

where G is the determinant of the induced metric G_{ab} computed on the constraints (2.3) and K^2 is a parameter which will be fixed later.² For our NR system, $\sqrt{-G}$ is given by

$$\sqrt{-G} = \frac{R^2\alpha^2}{\alpha^2 - \beta^2} [(\kappa^2 - \omega_1^2) + (\omega_1^2 - \omega_2^2) \cos^2 \theta]. \quad (3.9)$$

We want the field ϕ , defined in (3.8) through NR quantities, to *identically* satisfy (3.7) derived from the CSG equations. This imposes relations between the parameters involved, which are given in appendix A. In this way, we mapped *all* string solutions on $R_t \times S^3$ (in particular on $R_t \times S^2$) described by the NR integrable system onto solutions of the CSG (in particular SG) equations. From (A.1) one can see that the parameters A and C_χ are nonzero in general on $R_t \times S^2$ where $\omega_2 = C_2 = 0$. This means that there exist string solutions on $R_t \times S^2$ which correspond to solutions of the CSG system. Only when $M^2 = \kappa^2$, all string solutions on $R_t \times S^2$ are represented by solutions of the SG equation.

For the GM and SS solutions, which we are interested in, the relations between the NR and CSG parameters simplify a lot. Let us write them explicitly. The GM solutions correspond to $C_2 = 0$, $\kappa^2 = \omega_1^2$. This leads to

$$C_\phi = \frac{2}{\alpha^2 - \beta^2} \left[3M^2 - 2 \left(\omega_1^2 - \frac{\omega_2^2}{1 - \beta^2/\alpha^2} \right) \right], \quad K^2 = R^2 M^2, \quad (3.10)$$

$$A^2 = \frac{4}{\alpha^2/\beta^2 - 1} \left(M^2 - \omega_1^2 + \frac{\omega_2^2}{1 - \beta^2/\alpha^2} \right), \quad C_\chi = 0.$$

Therefore, for all GM strings the field χ is linear function at most. Since $A = C_\chi = 0$ implies $\chi = 0$, it follows from here that there exist GM string solutions on $R_t \times S^3$, which are mapped not on CSG solutions but on SG solutions instead. This happens exactly when

$$M^2 = \omega_1^2 - \frac{\omega_2^2}{1 - \beta^2/\alpha^2}.$$

²For $K^2 = \kappa^2$, this definition of the angle ϕ coincides with the one used in [17], which is based on the Pohlmeyer's reduction procedure [37].

In that case the nonzero parameters are

$$K^2 = R^2 \left(\omega_1^2 - \frac{\omega_2^2}{1 - \beta^2/\alpha^2} \right), \quad C_\phi = \frac{2}{\alpha^2 - \beta^2} \left(\omega_1^2 - \frac{\omega_2^2}{1 - \beta^2/\alpha^2} \right),$$

and the corresponding solution of the SG equation can be found from (3.7) to be

$$\sin(\phi/2) = \frac{1}{\cosh \left[\sqrt{\frac{\omega_1^2 - \omega_2^2/(1 - \beta^2/\alpha^2)}{1 - \beta^2/\alpha^2}} \left(\sigma + \frac{\beta}{\alpha} \tau \right) - \eta_0 \right]}, \quad \eta_0 = \text{const.} \quad (3.11)$$

Replacing (3.11) in (3.8), (3.9), one obtains the GM solution (A.4) as it should be.

For the SS solutions $C_2 = 0$, $\kappa^2 = \omega_1^2 \alpha^2 / \beta^2$. This results in

$$\begin{aligned} C_\phi &= \frac{2}{\beta^2 - \alpha^2} \left[2 \left(2\omega_1^2 \alpha^2 / \beta^2 + \frac{\omega_2^2}{\beta^2/\alpha^2 - 1} \right) - 3M^2 \right], \\ A^2 &= \frac{4}{M^4(1 - \alpha^2/\beta^2)} (\omega_1^2 \alpha^2 / \beta^2 - M^2)^2 \left(\frac{\omega_2^2}{\beta^2/\alpha^2 - 1} - M^2 \right), \\ C_\chi &= \frac{2\omega_1^2 \omega_2 \alpha^3}{M^2(\beta^2 - \alpha^2)\beta^2}, \quad K^2 = R^2 M^2. \end{aligned} \quad (3.12)$$

We want to point out that C_χ is always nonzero on S^3 contrary to the GM case, which makes χ also non-vanishing. To our knowledge, the CSG solutions corresponding to the SS on $R_t \times S^3$ are not given in the literature. To study this problem, we will consider the case when $A = 0$. A can be zero when

$$M^2 = \kappa^2 = \omega_1^2 \alpha^2 / \beta^2 \quad \text{or} \quad M^2 = \frac{\omega_2^2}{\beta^2/\alpha^2 - 1}, \quad (3.13)$$

As is seen from (3.13), we have two options, and we restrict ourselves to the first one.³ Replacing $M^2 = \omega_1^2 \alpha^2 / \beta^2$ in (3.12) and using the resulting expressions for C_ϕ and C_χ in (3.7), one obtains the simplified equation

$$\phi'^2 = \frac{4}{\beta^2 - \alpha^2} \left[\omega_1^2 \frac{\alpha^2}{\beta^2} \cos^2(\phi/2) - \frac{\omega_2^2}{\beta^2/\alpha^2 - 1} \cot^2(\phi/2) \right]$$

with solution

$$\sin^2(\phi/2) = \tanh^2(C\xi) + \frac{\omega_2^2}{\omega_1^2(1 - \alpha^2/\beta^2) \cosh^2(C\xi)}, \quad (3.14)$$

where

$$C = \frac{\alpha\omega_1 \sqrt{1 - \alpha^2/\beta^2 - \omega_2^2/\omega_1^2}}{\beta^2(1 - \alpha^2/\beta^2)}.$$

This agrees with eqs. (3.8) and (3.9). By inserting (3.14) into (3.5) one can find

$$\chi = \tilde{\chi} = 2 \arctan \left[\frac{\omega_1}{\omega_2} \sqrt{1 - \alpha^2/\beta^2 - \omega_2^2/\omega_1^2} \tanh(C\xi) \right].$$

³It turns out that the second option does not allow real solutions.

Hence, the CSG field ψ for the case at hand is given by

$$\psi = \sqrt{\tanh^2(C\xi) + \frac{\omega_2^2}{\omega_1^2(1 - \alpha^2/\beta^2)\cosh^2(C\xi)}} \quad (3.15)$$

$$\times \exp \left\{ i \arctan \left[\frac{\omega_1}{\omega_2} \sqrt{1 - \alpha^2/\beta^2 - \omega_2^2/\omega_1^2} \tanh(C\xi) \right] \right\}.$$

Here we have set the integration constants ϕ_0, χ_0 equal to zero. Several examples, which illustrate the established NR - CSG correspondence, are considered in the appendix A.

4. Finite-Size effects

In this section, we will obtain finite-size string solutions, their images in the (complex) sine-Gordon system, and the leading corrections to the SS “ $E - \Delta\varphi$ ” relation: first for the $R_t \times S^2$ case, then for the SS string with two angular momenta.

4.1 Strings on $R_t \times S^2$

Here we present the solutions of eq. (A.3), when

$$0 < \frac{\kappa^2}{\omega_1^2} < 1, \quad 0 < \frac{\beta^2 \kappa^2}{\alpha^2 \omega_1^2} < 1$$

for the two possibilities: $\alpha^2 > \beta^2$ and $\alpha^2 < \beta^2$. The first case reduces to the GM string in the limit $\kappa^2 = \omega_1^2$, while the second one corresponds to the SS solution in $\alpha^2 \omega_1^2 = \beta^2 \kappa^2$ limit.

4.1.1 The giant magnon

For $\alpha^2 > \beta^2$ the solution of eq. (A.3) is given by

$$\cos \theta = \frac{\cos \theta_{\min}}{dn(C(\xi - \xi_0)|m)}, \quad \cos \theta_{\min} \equiv \sqrt{1 - \kappa^2/\omega_1^2}, \quad (4.1)$$

$$C = \mp \frac{\omega_1 \sqrt{1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2}}{\alpha(1 - \beta^2 / \alpha^2)}, \quad m \equiv \frac{\kappa^2(1 - \beta^2 / \alpha^2)}{\omega_1^2(1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2)},$$

where $dn(u|m)$ is one of the elliptic functions and ξ_0 is an integration constant. The modulus m is positive. By using the relation [40]

$$dn(u + \mathbf{K}(m)|m) = \frac{\sqrt{1 - m}}{dn(u|m)},$$

and after choosing $C\xi_0 = \mathbf{K}$, the above solution can be rewritten as

$$\cos \theta = \cos \theta_{\max} dn(C\xi|m), \quad \cos \theta_{\max} \equiv \sqrt{1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2}. \quad (4.2)$$

In this form, (4.2) corresponds to the Arutyunov-Frolov-Zamaklar solution [30] (see also [31, 34]).

Inserting (4.1) into (2.15), one can find [39]

$$f_1 = \frac{\beta(\omega_1^2 - \kappa^2)}{\omega_1(\alpha^2 - \beta^2)} \left[\xi + \frac{\sqrt{\alpha^2 - \beta^2}}{\omega_1^2} \Pi \left(am(C\xi - \mathbf{K}), -\frac{m}{\kappa^2} \middle| m \right) \right],$$

where Π is the elliptic integral of third kind. Hence, the string solution is given by

$$\begin{aligned} W_1 &= R \sqrt{1 - (1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2) dn^2(C\xi|m)} \\ &\quad \times \exp \left\{ \frac{i\omega_1}{1 - \beta^2 / \alpha^2} \left[(1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2) \tau + (1 - \kappa^2 / \omega_1^2) \frac{\beta}{\alpha} \sigma \right] \right. \\ &\quad \left. + \frac{i\beta(1 - \kappa^2 / \omega_1^2)}{\alpha \omega_1 \sqrt{1 - \beta^2 / \alpha^2}} \Pi \left(am(C\xi - \mathbf{K}), -\frac{m}{\kappa^2} \middle| m \right) \right\}, \\ W_2 &= R \sqrt{1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2} dn(C\xi|m), \quad Z_0 = R \exp(i\kappa\tau). \end{aligned} \quad (4.3)$$

Now, let us see which solution of the SG equation is the image of (4.3). From eqs. (3.8) and (3.9), we have

$$\sin^2(\phi/2) = \frac{\kappa^2 (1 - \kappa^2 / \omega_1^2) sn^2(C\xi - \mathbf{K}|m)}{M^2 (1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2) dn^2(C\xi - \mathbf{K}|m)}. \quad (4.4)$$

This solution of the CSG system reduces to that of the SG equation for $M^2 = \kappa^2$. On the other hand, eq. (3.7) with $M^2 = \kappa^2$ gives

$$\sin^2(\phi/2) = sn^2 \left(\mp \frac{\kappa \sqrt{\omega_1^2 / \kappa^2 - 1}}{\alpha (1 - \beta^2 / \alpha^2)} (\xi - \xi_0) \middle| -\frac{1 - \beta^2 / \alpha^2}{\omega_1^2 / \kappa^2 - 1} \right).$$

One can see that the two results match if $M^2 = \kappa^2$ and $C\xi_0 = \mathbf{K}$ from an identity [40]

$$sn(u|m) = \frac{1}{\sqrt{1+m}} \frac{sn(u\sqrt{1+m}|m(1+m)^{-1})}{dn(u\sqrt{1+m}|m(1+m)^{-1})}.$$

In order to find the energy-charge relation for this string configuration, we need first to compute the conserved quantities. In accordance with (2.11), we have

$$\begin{aligned} \mathcal{E}_s &\equiv \frac{2\pi}{\sqrt{\lambda}} E_s = -2 \frac{\kappa}{\alpha} \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\theta}{\theta'} = 2 \frac{\kappa(1 - \beta^2 / \alpha^2)}{\omega_1 \sqrt{1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2}} \mathbf{K}(m), \\ \mathcal{J} &\equiv \frac{2\pi}{\sqrt{\lambda}} J_1 = 2 \frac{\alpha \omega_1}{\alpha^2 - \beta^2} \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\theta}{\theta'} \left(\sin^2 \theta - \frac{\beta^2 \kappa^2}{\alpha^2 \omega_1^2} \right) = 2 \sqrt{1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2} [\mathbf{K}(m) - \mathbf{E}(m)], \end{aligned}$$

which leads to

$$\mathcal{E}_s - \mathcal{J} = 2 \sqrt{1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2} \left[\mathbf{E}(m) - (1 - \kappa / \omega_1) \frac{1 + \beta^2 \kappa / \alpha^2 \omega_1}{1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2} \mathbf{K}(m) \right].$$

The worldsheet momentum can be expressed as

$$p = 2 \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\theta}{\theta'} f'_1 = -2 \frac{\beta / \alpha}{\sqrt{1 - \beta^2 \kappa^2 / \alpha^2 \omega_1^2}} \left[\frac{\alpha^2}{\beta^2} \Pi \left(1 - \frac{\alpha^2}{\beta^2} \middle| m \right) - \mathbf{K}(m) \right].$$

In the above expressions, $\mathbf{K}(m)$, $\mathbf{E}(m)$ and $\Pi(n|m)$ are the complete elliptic integrals.

In terms of new parameters defined by

$$\epsilon \equiv 1 - m, \quad v \equiv -\beta/\alpha,$$

these expressions can be simplified as follows:

$$\begin{aligned} \mathcal{E}_s &= 2\sqrt{(1-v^2)(1-\epsilon)}\mathbf{K}(1-\epsilon), & \mathcal{J} &= 2\sqrt{\frac{1-v^2}{1-v^2\epsilon}}[\mathbf{K}(1-\epsilon) - \mathbf{E}(1-\epsilon)], \\ \mathcal{E}_s - \mathcal{J} &= 2\sqrt{\frac{1-v^2}{1-v^2\epsilon}}\left[\mathbf{E}(1-\epsilon) - \left(1 - \sqrt{(1-v^2\epsilon)(1-\epsilon)}\right)\mathbf{K}(1-\epsilon)\right], \\ p &= 2v\sqrt{\frac{1-v^2\epsilon}{1-v^2}}\left[\frac{1}{v^2}\Pi\left(1 - \frac{1}{v^2}|1-\epsilon\right) - \mathbf{K}(1-\epsilon)\right]. \end{aligned}$$

We are interested in the behavior of these quantities in the limit $\epsilon \rightarrow 0$. To establish it, we will use the expansions for the elliptic functions given in appendix B.

Our approach is as follows. First, we expand \mathcal{E}_s , \mathcal{J} and p about $\epsilon = 0$ keeping v independent of ϵ . Second, we introduce $v(\epsilon)$ according to the rule

$$v(\epsilon) = v_0(p) + v_1(p)\epsilon + v_2(p)\epsilon \log(\epsilon)$$

and expand again. For p to be finite, we find

$$\begin{aligned} v_0(p) &= \cos(p/2), & v_1(p) &= \frac{1}{4}\sin^2(p/2)\cos(p/2)(1 - \log(16)), \\ v_2(p) &= \frac{1}{4}\sin^2(p/2)\cos(p/2). \end{aligned}$$

After that, from the expansion for \mathcal{J} , we obtain ϵ as a function of \mathcal{J} and p

$$\epsilon = 16 \exp\left(-\frac{\mathcal{J}}{\sin(p/2)} - 2\right).$$

Finally, using all these in the expansion for $\mathcal{E}_s - \mathcal{J}$, we derive

$$\mathcal{E}_s - \mathcal{J} = 2\sin(p/2)\left[1 - 4\sin^2(p/2)\exp\left(-\frac{\mathcal{J}}{\sin(p/2)} - 2\right)\right],$$

which reproduces the leading finite-size effects of the GM derived in [30, 31, 34].

4.1.2 The single spike

Now, we are going to consider the second possibility, namely, $\alpha^2 < \beta^2$. This time, the solution of the equation (A.3) can be written as

$$\begin{aligned} \cos\theta &= \cos\theta_{\max}dn(C\xi|m), & \cos\theta_{\max} &\equiv \sqrt{1 - \kappa^2/\omega_1^2}, \\ C &= \pm \frac{\alpha\omega_1\sqrt{1 - \kappa^2/\omega_1^2}}{\beta^2(1 - \alpha^2/\beta^2)}, & m &\equiv \frac{\beta^2/\alpha^2 - 1}{\omega_1^2/\kappa^2 - 1}. \end{aligned} \tag{4.5}$$

Here the new modulus m is positive again. From (2.15), one can find

$$f_1 = \pm \frac{\beta/\alpha}{\sqrt{1-\kappa^2/\omega_1^2}} \Pi(am(C\xi), \beta^2/\alpha^2 - 1|m) - \frac{\omega_1}{1-\alpha^2/\beta^2} \left(\frac{\alpha}{\beta} \sigma + \tau \right).$$

This results in the following string solution

$$\begin{aligned} W_1 &= R \sqrt{1 - (1 - \kappa^2/\omega_1^2) dn^2(C\xi|m)} \\ &\times \exp \left\{ -i\omega_1 \frac{\alpha/\beta}{1-\alpha^2/\beta^2} \left(\sigma + \frac{\alpha}{\beta} \tau \right) \pm i \frac{\beta/\alpha}{\sqrt{1-\kappa^2/\omega_1^2}} \Pi(am(C\xi), \beta^2/\alpha^2 - 1|m) \right\}, \\ W_2 &= R \sqrt{1 - \kappa^2/\omega_1^2} dn(C\xi|m), \quad Z_0 = R \exp(i\kappa\tau). \end{aligned}$$

From eqs. (3.8) and (3.9), the CSG solution corresponding to (4.6)

$$\sin^2(\phi/2) = \frac{\kappa^2}{M^2} sn^2(C\xi|m) \tag{4.6}$$

becomes that of the SG after fixing $M^2 = \kappa^2$. On the other hand, from (3.7) we get

$$\sin^2(\phi/2) = sn^2 \left(\pm \frac{\kappa \sqrt{\omega_1^2/\kappa^2 - 1}}{\alpha(\beta^2/\alpha^2 - 1)} (\xi - \xi_0) \middle| m \right).$$

Setting $\xi_0 = 0$ and rewriting

$$\pm \frac{\kappa \sqrt{\omega_1^2/\kappa^2 - 1}}{\alpha(\beta^2/\alpha^2 - 1)} \xi = \pm \omega_1 \frac{\alpha \sqrt{1 - \kappa^2/\omega_1^2}}{\beta(1 - \alpha^2/\beta^2)} \left(\frac{\alpha}{\beta} \sigma + \tau \right) = C\xi,$$

we find agreement with (4.6) if $M^2 = \kappa^2$.

Next, let us compute the conserved quantities for the present string solution. By using (2.11), we receive

$$\begin{aligned} \mathcal{E}_s &= 2 \frac{\kappa}{\alpha} \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\theta}{\theta'} = 2 \frac{\kappa(\beta^2/\alpha^2 - 1)}{\omega_1 \sqrt{1 - \kappa^2/\omega_1^2}} \mathbf{K}(m), \\ \mathcal{J} &= \frac{2}{\alpha} \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\theta}{\theta'} \sin^2 \theta (\beta f'_1 + \omega_1) = 2 \sqrt{1 - \kappa^2/\omega_1^2} \left[\mathbf{E}(m) - \frac{1 - \beta^2 \kappa^2/\alpha^2 \omega_1^2}{1 - \kappa^2/\omega_1^2} \mathbf{K}(m) \right]. \end{aligned}$$

In addition, we compute $\Delta\varphi_1$

$$\Delta\varphi \equiv \Delta\varphi_1 = 2 \int_{\theta_{\min}}^{\theta_{\max}} \frac{d\theta}{\theta'} f'_1 = -2 \frac{\beta/\alpha}{\sqrt{1 - \kappa^2/\omega_1^2}} \left[\Pi \left(1 - \frac{\beta^2}{\alpha^2} \middle| m \right) - \mathbf{K}(m) \right].$$

Defining parameters

$$\epsilon \equiv 1 - m, \quad v \equiv \beta/\alpha,$$

we can rewrite these as

$$\begin{aligned} \mathcal{E}_s &= 2 \sqrt{(v^2 - 1)(1 - \epsilon)} \mathbf{K}(1 - \epsilon), \quad \mathcal{J} = 2 \sqrt{\frac{v^2 - 1}{v^2 - \epsilon}} [\mathbf{E}(1 - \epsilon) - \epsilon \mathbf{K}(1 - \epsilon)], \\ \Delta\varphi &= -2v \sqrt{\frac{v^2 - \epsilon}{v^2 - 1}} [\Pi(1 - v^2|1 - \epsilon) - \mathbf{K}(1 - \epsilon)] \\ \mathcal{E}_s - \Delta\varphi &= 2v \sqrt{\frac{v^2 - \epsilon}{v^2 - 1}} \left[\Pi(1 - v^2|1 - \epsilon) - \left(1 - \frac{(v^2 - 1)\sqrt{1 - \epsilon}}{v\sqrt{v^2 - \epsilon}} \right) \mathbf{K}(1 - \epsilon) \right]. \end{aligned}$$

We proceed as in the GM case to find the ϵ expansion of these quantities. The difference is that now we impose \mathcal{J} to remain finite, which gives

$$\mathcal{J} = 2\sqrt{1 - \frac{1}{v_0^2}}, \quad v_1 = \frac{(v_0^2 - 1) [v_0^2(1 + \log(16)) - 2]}{4v_0}, \quad v_2 = -\frac{v_0(v_0^2 - 1)}{4}.$$

From the expansion for $\Delta\varphi$, we obtain ϵ as a function of $\Delta\varphi$ and \mathcal{J}

$$\epsilon = 16 \exp\left(-\frac{\sqrt{4 - \mathcal{J}^2}}{\mathcal{J}} \left[\Delta\varphi + \arcsin\left(\frac{\mathcal{J}}{2}\sqrt{4 - \mathcal{J}^2}\right)\right]\right).$$

Using these results in the expansion for $\mathcal{E}_s - \Delta\varphi$, one can see that the divergent terms cancel each other for $\mathcal{J}^2 < 2$ and the finite result is

$$\begin{aligned} E_s - \frac{\sqrt{\lambda}}{2\pi}\Delta\varphi &= \frac{\sqrt{\lambda}}{\pi} \left[\frac{1}{2} \arcsin\left(\frac{\mathcal{J}}{2}\sqrt{4 - \mathcal{J}^2}\right) + \frac{\mathcal{J}^3}{16\sqrt{4 - \mathcal{J}^2}} \epsilon \right] \\ &= \frac{\sqrt{\lambda}}{\pi} \left[\frac{p}{2} + 4 \sin^2 \frac{p}{2} \tan \frac{p}{2} \exp\left(-\frac{\Delta\varphi + p}{\tan \frac{p}{2}}\right) \right], \end{aligned} \quad (4.7)$$

where we used the identification [24, 36]

$$\arcsin(\mathcal{J}/2) = \frac{p}{2} = \bar{\theta} = \pi/2 - \arcsin \frac{\kappa}{\omega_1}.$$

This is the leading finite-size correction to the SS “ $E - \Delta\varphi$ ” relation. Let us also note that to the leading order, the length L of this SS string can be computed to be

$$L = \frac{\alpha}{\kappa} (\Delta\varphi + p).$$

4.2 Strings on $R_t \times S^3$

In the case $C_2 = 0$, eq. (2.13) can be written as

$$(\cos \theta)' = \mp \frac{\alpha\sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2} \sqrt{(z_+^2 - \cos^2 \theta)(\cos^2 \theta - z_-^2)}, \quad (4.8)$$

where

$$\begin{aligned} z_{\pm}^2 &= \frac{1}{2(1 - \frac{\omega_2^2}{\omega_1^2})} \left\{ y_1 + y_2 - \frac{\omega_2^2}{\omega_1^2} \pm \sqrt{(y_1 - y_2)^2 - \left[2(y_1 + y_2 - 2y_1y_2) - \frac{\omega_2^2}{\omega_1^2}\right] \frac{\omega_2^2}{\omega_1^2}} \right\}, \\ y_1 &= 1 - \kappa^2/\omega_1^2, \quad y_2 = 1 - \beta^2\kappa^2/\alpha^2\omega_1^2. \end{aligned}$$

The solution of (4.8) can be obtained as

$$\cos \theta = z_+ dn(C\xi|m), \quad C = \mp \frac{\alpha\sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2} z_+, \quad m \equiv 1 - z_-^2/z_+^2. \quad (4.9)$$

The solutions of eqs. (2.15) and (2.16) now read

$$\begin{aligned} f_1 &= \frac{2\beta/\alpha}{z_+\sqrt{1 - \omega_2^2/\omega_1^2}} \left[F(am(C\xi)|m) - \frac{\kappa^2/\omega_1^2}{1 - z_+^2} \Pi\left(am(C\xi), -\frac{z_+^2 - z_-^2}{1 - z_+^2} | m \right) \right], \\ f_2 &= \frac{2\beta\omega_2/\alpha\omega_1}{z_+\sqrt{1 - \omega_2^2/\omega_1^2}} F(am(C\xi)|m). \end{aligned}$$

Therefore, the full string solution is given by

$$\begin{aligned}
 Z_0 &= R \exp(i\kappa\tau), \\
 W_1 &= R \sqrt{1 - z_+^2 dn^2(C\xi|m)} \exp \left\{ i\omega_1\tau + \frac{2i\beta/\alpha}{z_+ \sqrt{1 - \omega_2^2/\omega_1^2}} \right. \\
 &\quad \times \left. \left[F(am(C\xi)|m) - \frac{\kappa^2/\omega_1^2}{1 - z_+^2} \Pi \left(am(C\xi), -\frac{z_+^2 - z_-^2}{1 - z_+^2} |m \right) \right] \right\}, \\
 W_2 &= R z_+ dn(C\xi|m) \exp \left\{ i\omega_2\tau + \frac{2i\beta\omega_2/\alpha\omega_1}{z_+ \sqrt{1 - \omega_2^2/\omega_1^2}} F(am(C\xi)|m) \right\}. \quad (4.10)
 \end{aligned}$$

We note that (4.10) contains both cases: $\alpha^2 > \beta^2$ for the GM and $\alpha^2 < \beta^2$ for the SS.

To find the CSG solution related to (4.10), we insert (4.9) into (3.8) and (3.9) to get

$$\sin^2(\phi/2) = \frac{\omega_1^2/M^2}{\beta^2/\alpha^2 - 1} \left[(1 - \kappa^2/\omega_1^2) - (1 - \omega_2^2/\omega_1^2) (z_+^2 cn^2(C\xi|m) + z_-^2 sn^2(C\xi|m)) \right] \quad (4.11)$$

After that, we use (4.11) in (3.5) and integrate. The result is

$$\chi = \frac{A}{\beta}(\beta\sigma + \alpha\tau) - C_\chi(\alpha\sigma + \beta\tau) + \frac{C_\chi}{CD} \Pi(am(C\xi), n|m), \quad (4.12)$$

where A/β and C_χ are given in (A.1), $C_2 = 0$, and

$$D = \frac{\omega_1^2/M^2}{\beta^2/\alpha^2 - 1} \left[(1 - \kappa^2/\omega_1^2) - (1 - \omega_2^2/\omega_1^2) z_+^2 \right], \quad n = \frac{(1 - \omega_2^2/\omega_1^2) (z_+^2 - z_-^2)}{(1 - \kappa^2/\omega_1^2) - (1 - \omega_2^2/\omega_1^2) z_+^2}.$$

Hence for the present case, the CSG field $\psi = \sin(\phi/2) \exp(i\chi/2)$ is defined by (4.11) and (4.12).

In a recent paper [41], the finite-size effects for dyonic GM have been considered and the leading order correction to the $\mathcal{E}_s - \mathcal{J}_1$ relation has been found to be

$$\begin{aligned}
 \mathcal{E}_s - \mathcal{J}_1 &= \sqrt{\mathcal{J}_2^2 + 4 \sin^2(p_1/2)} \\
 &\quad - 8 \frac{\sin^3(p_1/2)}{\cosh(\tilde{\theta}/2)} \exp \left[-\frac{2 \sin^2(p_1/2) \cosh^2(\tilde{\theta}/2)}{\sin^2(p_1/2) + \sinh^2(\tilde{\theta}/2)} \left(\frac{\mathcal{J}_1/2}{\sin(p_1/2) \cosh(\tilde{\theta}/2)} + 1 \right) \right],
 \end{aligned}$$

where

$$\cosh(\tilde{\theta}/2) = \frac{\sqrt{(\mathcal{J}_2/2)^2 + \sin^2(p_1/2)}}{\sin(p_1/2)}.$$

Our concern here is the SS for the case $\alpha^2 < \beta^2$ with two angular momenta \mathcal{J}_1 and \mathcal{J}_2 . The computation of the conserved quantities (2.11) and $\Delta\varphi_1$ now gives

$$\begin{aligned}
 \mathcal{E}_s &= \frac{2\kappa(\beta^2/\alpha^2 - 1)}{\omega_1 \sqrt{1 - \omega_2^2/\omega_1^2} z_+} \mathbf{K} \left(1 - z_-^2/z_+^2 \right), \\
 \mathcal{J}_1 &= \frac{2z_+}{\sqrt{1 - \omega_2^2/\omega_1^2}} \left[\mathbf{E} \left(1 - z_-^2/z_+^2 \right) - \frac{1 - \beta^2\kappa^2/\alpha^2\omega_1^2}{z_+^2} \mathbf{K} \left(1 - z_-^2/z_+^2 \right) \right],
 \end{aligned}$$

$$\mathcal{J}_2 = -\frac{2z_+\omega_2/\omega_1}{\sqrt{1-\omega_2^2/\omega_1^2}}\mathbf{E}\left(1-z_-^2/z_+^2\right),$$

$$\Delta\varphi = -\frac{2\beta/\alpha}{\sqrt{1-\omega_2^2/\omega_1^2}z_+}\left[\frac{\kappa^2/\omega_1^2}{1-z_+^2}\Pi\left(-\frac{z_+^2-z_-^2}{1-z_+^2}\middle|1-z_-^2/z_+^2\right)-\mathbf{K}\left(1-z_-^2/z_+^2\right)\right].$$

Our next step is to introduce the new parameters

$$\epsilon \equiv z_-^2/z_+^2, \quad v \equiv \beta/\alpha, \quad u \equiv \omega_2^2/\omega_1^2,$$

and to rewrite $\mathcal{E}_s, \mathcal{J}_1, \mathcal{J}_2, \Delta\varphi$ in the form

$$\begin{aligned} \mathcal{E}_s &= 2K_e\mathbf{K}(1-\epsilon), \\ \mathcal{J}_1 &= 2K_{11}[\mathbf{E}(1-\epsilon)-K_{12}\mathbf{K}(1-\epsilon)], \\ \mathcal{J}_2 &= 2K_2\mathbf{E}(1-\epsilon), \\ \Delta\varphi &= 2K_{\varphi 1}[K_{\varphi 2}\Pi(K_{\varphi 3}|1-\epsilon)-\mathbf{K}(1-\epsilon)]. \end{aligned} \tag{4.13}$$

The explicit ϵ -expansions of the coefficients $K_e, \dots, K_{\varphi 3}$ are given as functions of u and v in appendix B.

We also need to consider the ϵ -expansion for u and v as follows:

$$v(\epsilon) = v_0 + v_1\epsilon + v_2\epsilon \log(\epsilon), \quad u(\epsilon) = u_0 + u_1\epsilon + u_2\epsilon \log(\epsilon).$$

The coefficients can be determined by the condition that \mathcal{J}_1 and \mathcal{J}_2 should be finite,

$$v_0 = \frac{2\mathcal{J}_1}{\sqrt{(\mathcal{J}_1^2 - \mathcal{J}_2^2)[4 - (\mathcal{J}_1^2 - \mathcal{J}_2^2)]}}, \quad u_0 = \frac{\mathcal{J}_2^2}{\mathcal{J}_1^2}, \tag{4.14}$$

$$\begin{aligned} v_1 &= \frac{(1-u_0)v_0^2 - 1}{4(u_0 - 1)(v_0^2 - 1)v_0} \left\{ (u_0 - 1)v_0^4(1 + \log(16)) - 2 \right. \\ &\quad \left. + v_0^2[3 + \log(16) + u_0(\log(4096) - 5)] \right\}, \end{aligned} \tag{4.15}$$

$$v_2 = -\frac{v_0[1 - (1-u_0)v_0^2][1 + 3u_0 - (1-u_0)v_0^2]}{4(1-u_0)(v_0^2 - 1)},$$

$$u_1 = \frac{u_0[1 - (1-u_0)v_0^2]\log(16)}{v_0^2 - 1}, \quad u_2 = -\frac{u_0[1 - (1-u_0)v_0^2]}{v_0^2 - 1}.$$

The parameter ϵ can be obtained from $\Delta\varphi$ as follows:

$$\epsilon = 16 \exp\left(-\frac{\sqrt{(1-u_0)v_0^2 - 1}}{v_0^2 - 1}\left[\Delta\varphi + \arcsin\left(\frac{2\sqrt{(1-u_0)v_0^2 - 1}}{(1-u_0)v_0^2}\right)\right]\right). \tag{4.16}$$

From eqs. (4.14), (4.15) and (4.16), $\mathcal{E}_s - \Delta\varphi$ can be derived as

$$\mathcal{E}_s - \Delta\varphi = \arcsin N(\mathcal{J}_1, \mathcal{J}_2) + 2(\mathcal{J}_1^2 - \mathcal{J}_2^2)\sqrt{\frac{4}{[4 - (\mathcal{J}_1^2 - \mathcal{J}_2^2)]} - 1} \tag{4.17}$$

$$\times \exp\left[-\frac{2(\mathcal{J}_1^2 - \mathcal{J}_2^2)N(\mathcal{J}_1, \mathcal{J}_2)}{(\mathcal{J}_1^2 - \mathcal{J}_2^2)^2 + 4\mathcal{J}_2^2}[\Delta\varphi + \arcsin N(\mathcal{J}_1, \mathcal{J}_2)]\right], \tag{4.18}$$

$$N(\mathcal{J}_1, \mathcal{J}_2) \equiv \frac{1}{2} [4 - (\mathcal{J}_1^2 - \mathcal{J}_2^2)] \sqrt{\frac{4}{[4 - (\mathcal{J}_1^2 - \mathcal{J}_2^2)]}} - 1. \quad (4.19)$$

Here $\mathcal{J}_1^2 - \mathcal{J}_2^2 < 2$ is assumed. Finally, by using the SS relation between the angular momenta

$$\mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + 4 \sin^2(p/2)},$$

we obtain ($-\pi/2 \leq p \leq \pi/2$)

$$E_s - \frac{\sqrt{\lambda}}{2\pi} \Delta\varphi = \frac{\sqrt{\lambda}}{\pi} \left[\frac{p}{2} + 4 \sin^2 \frac{p}{2} \tan \frac{p}{2} \exp \left(-\frac{\tan \frac{p}{2} (\Delta\varphi + p)}{\tan^2 \frac{p}{2} + \mathcal{J}_2^2 \csc^2 p} \right) \right]. \quad (4.20)$$

This is our final result for the leading finite-size correction to the “ $E - \Delta\varphi$ ” relation for the SS string with two angular momenta. It is obvious that for $\mathcal{J}_2 = 0$ (4.20) reduces to (4.7) as it should be.

5. Concluding remarks

In this paper, by using the reduction of the string dynamics on $R_t \times S^3$ to the NR integrable system, we gave an explicit mapping connecting the parameters of *all* string solutions described by this dynamical system and the parameters in the corresponding solutions of the complex sine-Gordon integrable model. In the framework of this NR approach, we found finite-size string solutions, their images in the (complex) sine-Gordon system, and the leading finite-size corrections to the single spike “ $E - \Delta\varphi$ ” relation: both for strings on $R_t \times S^2$ and $R_t \times S^3$ backgrounds. It is an important open question to compare our results on the finite-size effects of the single spikes with the Lüscher corrections obtained from the exact S -matrices. The classical scattering amplitude computed in [36] turns out to be not sufficient since its complete pole structure is not clear. We hope that our results in the classical limit provide a clue to figure out the exact quantum S -matrix for the single spikes.

The GM and SS energy-charge relations for strings on $R_t \times S^5$ are already known for the infinite case [18, 21]. This opens a possibility to find the finite-size effects for such generalized string configurations. We are convinced that the NR approach will be effective in this case too. An evident direction of further development is to consider string configurations in the AdS part of the full $AdS_5 \times S^5$ background. It is known [26] that the corresponding integrable system will be again of NR-type, but with indefinite signature. Another interesting case is strings with nonzero spins on both AdS_5 and S^5 part of the target space. In this case, we will have two NR-type systems. While the equations of motion for them will decouple, the variables of the two NR systems will be mixed in the constraints [26]. Thus, a new kind of problem will appear. Nevertheless, there may exist string configurations for which this problem is solvable as found in [21] for example.

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A. Relation between the NR system and CSG

A.1 Explicit relations between the parameters

In the general case, the relation between the parameters in the solutions of the NR and CSG integrable systems is given by

$$\begin{aligned}
 K^2 &= R^2 M^2, \quad C_\phi = \frac{2}{\alpha^2 - \beta^2} \left\{ 3M^2 - 2 \left[\kappa^2 + \frac{(\kappa^2 - \omega_1^2) - \omega_2^2}{1 - \beta^2/\alpha^2} \right] \right\}, \\
 \frac{1}{4} M^4 (\alpha^2 - \beta^2) \frac{A^2}{\beta^2} &= M^4 \left(M^2 - \kappa^2 + \frac{\omega_2^2}{1 - \beta^2/\alpha^2} \right) \\
 &\quad - \left(\frac{\kappa^2 - \omega_1^2}{1 - \beta^2/\alpha^2} \right) \left\{ M^4 + \left[M^2 - \left(\frac{\kappa^2 - \omega_1^2}{1 - \beta^2/\alpha^2} \right) \right] \left(\frac{\kappa^2 - \omega_1^2}{1 - \beta^2/\alpha^2} \right) \right. \\
 &\quad \left. - \left[2M^2 - \left(\frac{\kappa^2 - \omega_1^2}{1 - \beta^2/\alpha^2} \right) \right] \left(\kappa^2 - \frac{\omega_2^2}{1 - \beta^2/\alpha^2} \right) \right\} \quad (\text{A.1}) \\
 &\quad - \frac{(\omega_1^2 - \omega_2^2)}{\omega_1^2 (1 - \beta^2/\alpha^2)^3} \left\{ [M^2 (1 - \beta^2/\alpha^2) - \kappa^2] (\omega_1^2 - \omega_2^2) \check{C}_2^2 \right. \\
 &\quad \left. - [M^2 (1 - \beta^2/\alpha^2) - (\kappa^2 - \omega_1^2)] \left[2 \frac{\beta}{\alpha} \omega_2 \kappa^2 \check{C}_2 + (\kappa^2 - \omega_1^2) \left(\frac{\beta^2}{\alpha^2} \kappa^2 - \omega_1^2 \right) \right] \right\}, \\
 \frac{1}{4} M^4 (\alpha^2 - \beta^2) C_\chi^2 &= - \left(\frac{\kappa^2 - \omega_1^2}{1 - \beta^2/\alpha^2} \right)^2 \left[\kappa^2 - \frac{(\kappa^2 - \omega_1^2) + \omega_2^2}{1 - \beta^2/\alpha^2} \right] \\
 &\quad + \frac{(\omega_1^2 - \omega_2^2)}{\omega_1^2 (1 - \beta^2/\alpha^2)^3} \left\{ \kappa^2 (\omega_1^2 - \omega_2^2) \check{C}_2^2 - (\kappa^2 - \omega_1^2) \right. \\
 &\quad \left. \times \left[2 \frac{\beta}{\alpha} \omega_2 \kappa^2 \check{C}_2 + (\kappa^2 - \omega_1^2) \left(\frac{\beta^2}{\alpha^2} \kappa^2 - \omega_1^2 \right) \right] \right\}, \quad (\text{A.2})
 \end{aligned}$$

where $\check{C}_2 = C_2/\alpha$. Thus, we have expressed the CSG parameters C_ϕ , A and C_χ through the NR parameters α , β , κ , ω_1 , ω_2 , C_2 . The mass parameter M remains free.

Let us consider several examples, which illustrate the established NR - CSG correspondence. We are interested in the GM and SS configurations on $R_t \times S^2$ and $R_t \times S^3$. From the NR-system viewpoint, we have to set $C_2 = 0$ in (2.13), (2.15) and (2.16) for the GM and SS string solutions. This condition is to require one of the turning points, where $\theta' = 0$, to lay on the equator of the sphere, i.e. $\theta = \pi/2$ [18].

A.2 On $R_t \times S^2$

We begin with the $R_t \times S^2$ case, when $C_2 = \omega_2 = 0$ and θ' in (2.13) takes the form

$$\theta' = \frac{\pm \alpha \omega_1}{(\alpha^2 - \beta^2) \sin \theta} \sqrt{\left(\frac{\beta^2 \kappa^2}{\alpha^2 \omega_1^2} - \sin^2 \theta \right) \left(\sin^2 \theta - \frac{\kappa^2}{\omega_1^2} \right)}. \quad (\text{A.3})$$

A.2.1 The giant magnon

The GM solution corresponds to $\kappa^2 = \omega_1^2$ with $\alpha^2 > \beta^2$, which is given by

$$\cos \theta = \frac{\sqrt{1 - \beta^2/\alpha^2}}{\cosh \left(\omega_1 \frac{\sigma + \tau \beta/\alpha}{\sqrt{1 - \beta^2/\alpha^2}} \right)}.$$

From (2.15), one finds $f_2 = 0$ and

$$f_1 = \arctan \left[\frac{\alpha}{\beta} \sqrt{1 - \beta^2/\alpha^2} \tanh \left(\omega_1 \frac{\sigma + \tau\beta/\alpha}{\sqrt{1 - \beta^2/\alpha^2}} \right) \right].$$

For $R^2 = M^2 = \omega_1^2 = 1$, $\beta/\alpha = -\sin\theta_0$, this string solution coincides with the Hofman-Maldacena solution [15], and is equivalent to the solution in [17] for $R_t \times S^2$ after the identification $W_1 = Z_1 \exp(i\pi/2)$, $W_2 = Z_2$. Now, the parameters in (3.10) take the values

$$\begin{aligned} K^2 &= R^2 \omega_1^2 = 1, & M^2 &= \omega_1^2 = 1, \\ C_\phi &= \frac{2\omega_1^2}{\alpha^2 - \beta^2} = \frac{2}{\alpha^2 \cos^2 \theta_0}, & A &= C_\chi = 0, \end{aligned}$$

and from eqs. (3.7), (3.8) and (3.9)) the corresponding SG solution becomes

$$\sin(\phi/2) = \frac{1}{\cosh \left(\frac{\sigma - \tau \sin \theta_0}{\cos \theta_0} - \eta_0 \right)}.$$

This can be also obtained from (3.11) by setting $\omega_2 = 0$.

However, for $M^2 > \omega_1^2 = \kappa^2$, we have

$$A = 2\beta \sqrt{\frac{M^2 - \omega_1^2}{\alpha^2 - \beta^2}} \neq 0.$$

This case is related to the CSG system instead of the SG one. It is interesting to find the CSG solution associated with it. Using (3.7) again, we find

$$\sin(\phi/2) = \frac{\omega_1}{M \cosh \left(\omega_1 \frac{\sigma + \tau\beta/\alpha}{\sqrt{1 - \beta^2/\alpha^2}} - \eta_0 \right)}, \quad \chi = 2\sqrt{\frac{M^2 - \omega_1^2}{1 - \beta^2/\alpha^2}} \left(\frac{\beta}{\alpha} \sigma + \tau \right).$$

A.2.2 The single spike

The SS solution corresponds to $\beta^2 \kappa^2 = \alpha^2 \omega_1^2$. In this case, the expressions for θ and f_1 are

$$\cos \theta = \frac{\sqrt{1 - \alpha^2/\beta^2}}{\cosh(C\xi)}, \quad f_1 = -\omega_1(\sigma\alpha/\beta + \tau) + \arctan \left[\frac{\beta}{\alpha} \sqrt{1 - \alpha^2/\beta^2} \tanh(C\xi) \right],$$

and the corresponding string solution is

$$\begin{aligned} W_1 &= R \sqrt{1 - \frac{1 - \alpha^2/\beta^2}{\cosh^2(C\xi)}} \exp \left\{ -i\omega_1 \sigma \alpha / \beta + i \arctan \left[\frac{\beta}{\alpha} \sqrt{1 - \alpha^2/\beta^2} \tanh(C\xi) \right] \right\}, \\ W_2 &= \frac{R \sqrt{1 - \alpha^2/\beta^2}}{\cosh(C\xi)}, & Z_0 &= R \exp \left(i \frac{\alpha}{\beta} \omega_1 \tau \right), \end{aligned}$$

where we used a short notation

$$C\xi \equiv \omega_1 \frac{\alpha}{\beta} \frac{\sigma\alpha/\beta + \tau}{\sqrt{1 - \alpha^2/\beta^2}}.$$

The “dual” SG solution can be obtained from (3.14) by setting $\omega_2 = 0$. If we choose $R = 1$, $\alpha/\beta = \sin\theta_1$, $\omega_1 = -\cot\theta_1$, $\beta = 1$, the SS solution on $R_t \times S^2$ in [24] is reproduced.

A.3 On $R_t \times S^3$

A.3.1 The giant magnon

Let us continue with the $R_t \times S^3$ case, when $C_2 = 0$, $\omega_2 \neq 0$. First, we would like to establish the correspondence between the dyonic GM string solution [18] ($\kappa^2 = \omega_1^2$) to those found in [17]

$$Z_1 = \frac{1}{\sqrt{1+k^2}} \left\{ \tanh \left[\cos \alpha^D \left(\sigma \sqrt{1+k^2 \cos^2 \alpha^D} - k\tau \cos \alpha^D \right) \right] - ik \right\} \exp(i\tau),$$

$$Z_2 = \frac{1}{\sqrt{1+k^2}} \frac{\exp \left[i \sin \alpha^D \left(\tau \sqrt{1+k^2 \cos^2 \alpha^D} - k\sigma \cos \alpha^D \right) \right]}{\cosh \left[\cos \alpha^D \left(\sigma \sqrt{1+k^2 \cos^2 \alpha^D} - k\tau \cos \alpha^D \right) \right]},$$

where the parameter k is related to the soliton rapidity $\hat{\theta}$ through the equality

$$k = \frac{\sinh \hat{\theta}}{\cos \alpha^D},$$

and α^D determines the U(1) charge carried by the CSG soliton [17].

The solutions of eqs. (2.13), (2.15) and (2.16) are given by

$$\cos \theta = \frac{\cos \theta_0}{\cosh(C\xi)}, \quad f_1 = \arctan [\cot \theta_0 \tanh(C\xi)], \quad f_2 = \frac{\beta \omega_2}{\alpha^2 - \beta^2} \xi,$$

$$\sin^2 \theta_0 \equiv \frac{\beta^2 \omega_1^2}{\alpha^2 (\omega_1^2 - \omega_2^2)}, \quad C \equiv \frac{\alpha \sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2} \cos \theta_0.$$

Then, the comparison shows that the two solutions are equivalent if

$$Z_1 \exp(i\pi/2) = W_1 = R \sin \theta \exp [i (\omega_1 \tau + f_1)],$$

$$Z_2 = W_2 = R \cos \theta \exp [i (\omega_2 \tau + f_2)],$$

$$R = \kappa = \omega_1 = 1, \quad \alpha = \cos \alpha^D \sqrt{1+k^2 \cos^2 \alpha^D},$$

$$\beta = -k \cos^2 \alpha^D, \quad \omega_2 = \frac{\sin \alpha^D}{\sqrt{1+k^2 \cos^2 \alpha^D}}.$$

As a consequence, the CSG parameters in (3.10) reduce to

$$C_\phi = \frac{2}{\cos^2 \alpha^D} (1 + 2 \sin^2 \alpha^D), \quad A = k \sin(2\alpha^D), \quad C_\chi = 0, \quad K^2 = 1.$$

A.3.2 The single spike

Now, let us turn to the SS solutions on $R_t \times S^3$ as described by the NR integrable system [27]. By using the SS-condition $\beta^2 \kappa^2 = \alpha^2 \omega_1^2$ in (2.13) one derives

$$\theta' = \frac{\alpha \sqrt{\omega_1^2 - \omega_2^2}}{\alpha^2 - \beta^2} \frac{\cos \theta}{\sin \theta} \sqrt{\sin^2 \theta - \frac{\alpha^2 \omega_1^2}{\beta^2 (\omega_1^2 - \omega_2^2)}},$$

whose solution is given by

$$\cos \theta = \frac{\sqrt{(1 - \alpha^2/\beta^2) \omega_1^2 - \omega_2^2}}{\sqrt{\omega_1^2 - \omega_2^2} \cosh(C\xi)}, \quad C\xi \equiv \sqrt{\omega_1^2 - \frac{\omega_2^2}{1 - \alpha^2/\beta^2}} \frac{\alpha(\sigma\alpha/\beta + \tau)}{\sqrt{\beta^2 - \alpha^2}}.$$

By using (2.15), (2.16), one finds the following expressions for the string embedding coordinates $\varphi_j = \omega_j \tau + f_j$

$$\begin{aligned}\varphi_1 &= -\omega_1 \sigma \alpha / \beta + \arctan \left\{ \frac{\beta}{\alpha \omega_1} \sqrt{\left(1 - \frac{\alpha^2}{\beta^2}\right) \left(\omega_1^2 - \frac{\omega_2^2}{1 - \alpha^2 / \beta^2}\right)} \tanh(C\xi) \right\} \\ \varphi_2 &= -\omega_2 \frac{\alpha(\sigma + \tau \alpha / \beta)}{\beta(1 - \alpha^2 / \beta^2)}.\end{aligned}$$

Comparing the above results with the SS string solution given in (4.1)–(4.7) of [36], we see that the two solutions coincide for

$$R = 1, \quad \sin \theta_1 = -\frac{1}{\sqrt{\omega_1^2 - \omega_2^2}}, \quad \sin \gamma_1 = \frac{\omega_2}{\omega_1}, \quad \omega_1 = -\frac{\beta}{\alpha}. \quad (\text{A.4})$$

From (3.12), the CSG parameters are

$$\begin{aligned}C_\phi &= \frac{2}{\beta^2 (1 - \sin^2 \theta_1 \cos^2 \gamma_1)} \left[4 - 3M^2 + \frac{2 \cos^4 \gamma_1}{\sin^2 \gamma_1 (1 - \sin^2 \theta_1 \cos^2 \gamma_1)} \right], \quad K^2 = M^2, \\ A &= \frac{M^2 - 1}{M^2 \sqrt{1 - \sin^2 \theta_1 \cos^2 \gamma_1}} \sqrt{\frac{\cos^4 \gamma_1}{\sin^2 \gamma_1 (1 - \sin^2 \theta_1 \cos^2 \gamma_1)}} - M^2, \\ C_\chi &= -\frac{2 \sin \gamma_1}{M^2 \beta (1 - \sin^2 \theta_1 \cos^2 \gamma_1)}.\end{aligned}$$

Comparing (A.4) with (3.13), one sees that the solution found in [36] corresponds actually to $M^2 = 1$ which leads to $A = 0$. Hence, the “dual” CSG solution is of the type (3.15).

B. ϵ -expansions

We use the following expansions for the elliptic functions

$$\begin{aligned}\mathbf{K}(1 - \epsilon) &\propto -\frac{1}{2} \log \epsilon (1 + O(\epsilon)) + \log(4) (1 + O(\epsilon)), \\ \mathbf{E}(1 - \epsilon) &\propto 1 - \epsilon \left(\frac{1}{4} - \log(2) \right) (1 + O(\epsilon)) - \frac{\epsilon}{4} \log \epsilon (1 + O(\epsilon)) \\ \Pi(n|1 - \epsilon) &\propto \frac{\log \epsilon}{2(n - 1)} (1 + O(\epsilon)) + \frac{\sqrt{n} \log \left(\frac{1 + \sqrt{n}}{1 - \sqrt{n}} \right) - \log(16)}{2(n - 1)} (1 + O(\epsilon)).\end{aligned}$$

The expansions for the coefficients in (4.13) are given by

$$\begin{aligned}K_e &\propto \frac{v^2 - 1}{\sqrt{v^2(1 - u) - 1}} - \frac{v^2(1 - u)^2 - 1}{2\sqrt{v^2(1 - u) - 1}(1 - u)} \epsilon, \\ K_{11} &\propto \sqrt{\frac{v^2(1 - u) - 1}{v^2(1 - u)^2}} - \frac{\sqrt{v^2(1 - u) - 1} (1 + v^2(2u - 1))}{2v^3(v^2 - 1)(1 - u)^2} \epsilon, \\ K_{12} &\propto \left(1 - \frac{v^2 u}{v^2 - 1} \right) \epsilon,\end{aligned}$$

$$\begin{aligned}
 K_2 &\propto -\sqrt{\frac{(v^2(1-u)-1)u}{v^2(1-u)^2}} + \frac{\sqrt{\frac{(v^2(1-u)-1)u}{v^2(1-u)^2}}(1+v^2(2u-1))}{2v^2(v^2-1)(1-u)} \epsilon, \\
 K_{\varphi_1} &\propto -\frac{v}{\sqrt{1-1/v^2-u}} - \frac{1+v^2(2u-1)}{2(v^2-1)\sqrt{v^2(1-u)-1}(1-u)} \epsilon, \\
 K_{\varphi_2} &\propto 1-u + \frac{(1-v^2(1-u))u}{v^2-1} \epsilon, \\
 K_{\varphi_3} &\propto 1-v^2(1-u) + 2v^2u \left(1 - \frac{v^2u}{v^2-1}\right) \epsilon.
 \end{aligned}$$

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