



Finite-size giant magnons on η -deformed $AdS_5 \times S^5$



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ABSTRACT

We consider strings moving in the $R_t \times S^3_\eta$ subspace of the η -deformed $AdS_5 \times S^5$ and obtain a class of solutions depending on several parameters. They are characterized by the string energy and two angular momenta. Finite-size dyonic giant magnon belongs to this class of solutions. Further on, we restrict ourselves to the case of giant magnon with one nonzero angular momentum, and obtain the leading finite-size correction to the dispersion relation.

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1. Introduction

In the recent years important progress has been made in the field of AdS/CFT duality [1] (for overview see [2]). The main achievements are due to the discovery of integrable structures on both sides of the correspondence.

The most developed case is the correspondence between strings moving in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in four dimensions. The so-called γ -deformation of $AdS_5 \times S^5$ has been proposed in [3]. It was shown in [4] that this deformation is still integrable for real γ (known as β - or TsT -deformation).

A new integrable deformation of the type IIB $AdS_5 \times S^5$ superstring action, depending on one real parameter η , has been found recently in [5]. The bosonic part of the superstring sigma model Lagrangian on this η -deformed background was determined in [6]. Then the authors of [6] used it to compute the perturbative S-matrix of bosonic particles in the model.

Interesting new developments were made in [7]. There the spectrum of a string moving on η -deformed $AdS_5 \times S^5$ is considered. This is done by treating the corresponding worldsheet theory as integrable field theory.²

In particular, it was found that the dispersion relation for the infinite-size giant magnons [9] in this background, in the large string tension limit $g \rightarrow \infty$ is given by

$$E = \frac{2g\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}} \operatorname{arcsinh}\left(\tilde{\eta} \sin \frac{p}{2}\right), \quad (1.1)$$

where $\tilde{\eta}$ is related to the deformation parameter η according to

$$\tilde{\eta} = \frac{2\eta}{1-\eta^2}. \quad (1.2)$$

Here, we are going to extend the result (1.1) to the case of finite-size giant magnons.

The paper is organized as follows. In Section 2 we give the bosonic part of the string Lagrangian on η -deformed $AdS_5 \times S^5$ found in [6] and extract from it the background fields. Then in Section 3, we obtain the exact solutions for the finite-size dyonic giant magnon coordinates, the corresponding conserved charges and the angular difference along one of the isometric coordinates on the deformed sphere S^3_η .³ In Section 4 we find the dispersion relation for the giant magnons with one nonzero angular momentum, including the leading finite-size effect on it. Section 5 is devoted to our concluding remarks.

2. String Lagrangian and background fields

The bosonic part of the string Lagrangian \mathcal{L} on the η -deformed $AdS_5 \times S^5$ found in [6] is given by a sum of the Lagrangians \mathcal{L}_a and \mathcal{L}_s , for the AdS and sphere subspaces. Since there is nonzero B -field on both subspaces, which leads to the appearance of Wess–Zumino terms, these Lagrangians can be further decomposed as

$$\mathcal{L}_a = L_a^g + L_a^{WZ}, \quad \mathcal{L}_s = L_s^g + L_s^{WZ}, \quad (2.1)$$

where the superscript “ g ” is related to the dependence on the background metric. The explicit expressions for the Lagrangians in (2.1) are as follows [6]

³ This angular difference is identified with the momentum of the magnon excitations in the dual spin chain.

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² See also [8].

$$L_a^g = -\frac{T}{2} \gamma^{\alpha\beta} \left[-\frac{(1+\rho^2)\partial_\alpha t \partial_\beta t}{1-\tilde{\eta}^2 \rho^2} + \frac{\partial_\alpha \rho \partial_\beta \rho}{(1+\rho^2)(1-\tilde{\eta}^2 \rho^2)} \right. \\ \left. + \frac{\rho^2 \partial_\alpha \zeta \partial_\beta \zeta}{1+\tilde{\eta}^2 \rho^4 \sin^2 \zeta} + \frac{\rho^2 \cos^2 \zeta \partial_\alpha \psi_1 \partial_\beta \psi_1}{1+\tilde{\eta}^2 \rho^4 \sin^2 \zeta} \right. \\ \left. + \rho^2 \sin^2 \zeta \partial_\alpha \psi_2 \partial_\beta \psi_2 \right], \quad (2.2)$$

$$L_a^{WZ} = \frac{T}{2} \tilde{\eta} \epsilon^{\alpha\beta} \frac{\rho^4 \sin 2\zeta}{1+\tilde{\eta}^2 \rho^4 \sin^2 \zeta} \partial_\alpha \psi_1 \partial_\beta \zeta, \quad (2.3)$$

$$L_s^g = -\frac{T}{2} \gamma^{\alpha\beta} \left[\frac{(1-r^2)\partial_\alpha \phi \partial_\beta \phi}{1+\tilde{\eta}^2 r^2} + \frac{\partial_\alpha r \partial_\beta r}{(1-r^2)(1+\tilde{\eta}^2 r^2)} \right. \\ \left. + \frac{r^2 \partial_\alpha \xi \partial_\beta \xi}{1+\tilde{\eta}^2 r^4 \sin^2 \xi} + \frac{r^2 \cos^2 \xi \partial_\alpha \phi_1 \partial_\beta \phi_1}{1+\tilde{\eta}^2 r^4 \sin^2 \xi} \right. \\ \left. + r^2 \sin^2 \xi \partial_\alpha \phi_2 \partial_\beta \phi_2 \right], \quad (2.4)$$

$$L_s^{WZ} = -\frac{T}{2} \tilde{\eta} \epsilon^{\alpha\beta} \frac{r^4 \sin 2\xi}{1+\tilde{\eta}^2 r^4 \sin^2 \xi} \partial_\alpha \phi_1 \partial_\beta \xi, \quad (2.5)$$

where we introduced the notation

$$T = g\sqrt{1+\tilde{\eta}^2}. \quad (2.6)$$

Comparing (2.2)–(2.5) with the Polyakov string Lagrangian, one can extract the components of the background fields. They are given by

$$g_{tt} = -\frac{1+\rho^2}{1-\tilde{\eta}^2 \rho^2}, \quad g_{\rho\rho} = \frac{1}{(1+\rho^2)(1-\tilde{\eta}^2 \rho^2)}, \\ g_{\zeta\zeta} = \frac{\rho^2}{1+\tilde{\eta}^2 \rho^4 \sin^2 \zeta} \\ g_{\psi_1 \psi_1} = \frac{\rho^2 \cos^2 \zeta}{1+\tilde{\eta}^2 \rho^4 \sin^2 \zeta}, \quad g_{\psi_2 \psi_2} = \rho^2 \sin^2 \zeta, \\ b_{\psi_1 \zeta} = \tilde{\eta} \frac{\rho^4 \sin 2\zeta}{1+\tilde{\eta}^2 \rho^4 \sin^2 \zeta}. \quad (2.7)$$

$$g_{\phi\phi} = \frac{1-r^2}{1+\tilde{\eta}^2 r^2}, \quad g_{rr} = \frac{1}{(1-r^2)(1+\tilde{\eta}^2 r^2)}, \\ g_{\xi\xi} = \frac{r^2}{1+\tilde{\eta}^2 r^4 \sin^2 \xi} \\ g_{\phi_1 \phi_1} = \frac{r^2 \cos^2 \xi}{1+\tilde{\eta}^2 r^4 \sin^2 \xi}, \quad g_{\phi_2 \phi_2} = r^2 \sin^2 \xi, \\ b_{\phi_1 \xi} = -\tilde{\eta} \frac{r^4 \sin 2\xi}{1+\tilde{\eta}^2 r^4 \sin^2 \xi}. \quad (2.8)$$

Since we are going to consider giant magnon solutions, we restrict ourselves to the $R_t \times S_\eta^3$ subspace, which corresponds to the following choice in AdS_η

$$\rho = 0, \quad \zeta = 0, \quad \psi_1 = \psi_2 = 0 \Rightarrow b_{\psi_1 \zeta} = 0.$$

On S_η^5 we first introduce the angle $\tilde{\theta}$ in the following way

$$r = \sin \tilde{\theta},$$

which leads to

$$ds_{S_\eta^5}^2 = \frac{\cos^2 \tilde{\theta}}{1+\tilde{\eta}^2 \sin^2 \tilde{\theta}} d\phi^2 + \frac{d\tilde{\theta}^2}{1+\tilde{\eta}^2 \sin^2 \tilde{\theta}} \\ + \frac{\sin^2 \tilde{\theta}}{1+\tilde{\eta}^2 \sin^4 \tilde{\theta} \sin^2 \xi} d\xi^2 + \frac{\sin^2 \tilde{\theta} \cos^2 \xi}{1+\tilde{\eta}^2 \sin^4 \tilde{\theta} \sin^2 \xi} d\phi_1^2 \\ + \sin^2 \tilde{\theta} \sin^2 \xi d\phi_2^2,$$

$$b_{\phi_1 \xi} = -\tilde{\eta} \frac{\sin^4 \tilde{\theta} \sin 2\xi}{1+\tilde{\eta}^2 \sin^4 \tilde{\theta} \sin^2 \xi}.$$

Now, to go to S_η^3 , we can safely set $\phi = 0$, $\tilde{\theta} = \frac{\pi}{2}$ (we also exchange ϕ_1 and ϕ_2 and replace ξ with θ). Thus, the background seen by the string moving in the $R_t \times S_\eta^3$ subspace can be written as

$$g_{tt} = -1, \quad g_{\phi_1 \phi_1} = \sin^2 \theta, \quad g_{\phi_2 \phi_2} = \frac{\cos^2 \theta}{1+\tilde{\eta}^2 \sin^2 \theta}, \\ g_{\theta\theta} = \frac{1}{1+\tilde{\eta}^2 \sin^2 \theta}, \quad b_{\phi_2 \theta} = -\tilde{\eta} \frac{\sin 2\theta}{1+\tilde{\eta}^2 \sin^2 \theta}. \quad (2.9)$$

3. Exact results

Here and further on we will work in conformal gauge when $\gamma^{\alpha\beta} = \text{diag}(-1, 1)$ and the string Lagrangian and Virasoro constraints have the form

$$L_s = \frac{T}{2} (G_{00} - G_{11} + 2B_{01}), \quad (3.1)$$

$$G_{00} + G_{11} = 0, \quad G_{01} = 0. \quad (3.2)$$

Above

$$G_{\alpha\beta} = g_{MN} \partial_\alpha X^M \partial_\beta X^N, \quad B_{\alpha\beta} = b_{MN} \partial_\alpha X^M \partial_\beta X^N$$

are the fields induced on the string worldsheet. For the case under consideration

$$X^M = (t, \phi_1, \phi_2, \theta).$$

The corresponding nonzero components of g_{MN} and b_{MN} are given in (2.9).

Now, we impose the following ansatz for the string embedding

$$t(\tau, \sigma) = \kappa \tau, \quad \phi_i(\tau, \sigma) = \omega_i \tau + F_i(\xi), \\ \theta(\tau, \sigma) = \theta(\xi), \quad \xi = \alpha \sigma + \beta \tau, \quad i = 1, 2, \quad (3.3)$$

where τ and σ are the string world-sheet coordinates, $F_i(\xi)$, $\theta(\xi)$ are arbitrary functions of ξ , and $\kappa, \omega_i, \alpha, \beta$ are parameters.

Replacing (3.3) into (3.1) one finds the following solutions of the equations of motion for $\phi_i(\tau, \sigma)$ (we introduced the notation $\chi \equiv \cos^2 \theta$)

$$\phi_1(\tau, \sigma) = \omega_1 \tau + \frac{1}{\alpha^2 - \beta^2} \int d\xi \left(\frac{C_1}{1-\chi} + \beta \omega_1 \right), \quad (3.4)$$

$$\phi_2(\tau, \sigma) = \omega_2 \tau + \frac{1}{\alpha^2 - \beta^2} \\ \times \int d\xi \left[\frac{(1+\tilde{\eta}^2)C_2}{\chi} + \beta \omega_2 - \tilde{\eta}^2 C_2 \right], \quad (3.5)$$

where C_1, C_2 are integration constants.

By using (3.4), (3.5), one can show that the Virasoro constraints (3.2) take the form

$$\left(\frac{d\chi}{d\xi}\right)^2 = \frac{4\chi(1-\chi)[1+\tilde{\eta}^2(1-\chi)]}{(\alpha^2-\beta^2)^2} \times \left[(\alpha^2+\beta^2)\kappa^2 - \frac{C_1^2}{1-\chi} - C_2^2 \frac{1+\tilde{\eta}^2(1-\chi)}{\chi} - \alpha^2\omega_1^2(1-\chi) - \alpha^2\omega_2^2 \frac{\chi}{1+\tilde{\eta}^2(1-\chi)} \right], \quad (3.6)$$

$$\omega_1 C_1 + \omega_2 C_2 + \beta\kappa^2 = 0. \quad (3.7)$$

Next, we solve (3.7) with respect to C_1 and replace the solution into (3.6). The result is

$$\left(\frac{d\chi}{d\xi}\right)^2 = \frac{4}{(\alpha^2-\beta^2)^2} \alpha^2 \tilde{\eta}^2 \omega_1^2 (\chi_\eta - \chi)(\chi_p - \chi) \times (\chi - \chi_m)(\chi - \chi_n), \quad (3.8)$$

where

$$\chi_\eta + \chi_p + \chi_m + \chi_n = -\frac{\alpha^2[\omega_2^2 - \omega_1^2 + \tilde{\eta}^2(\kappa^2 - 3\omega_1^2)] + \tilde{\eta}^2\beta^2\kappa^2 + \tilde{\eta}^4 C_2^2}{\alpha^2 \tilde{\eta}^2 \omega_1^2}, \quad (3.9)$$

$$\begin{aligned} &\chi_p \chi_\eta + (\chi_p + \chi_\eta)\chi_n + \chi_m(\chi_p + \chi_\eta + \chi_n) \\ &= \frac{1}{\tilde{\eta}^2 \alpha^2 \omega_1^4} \{ \beta^2 \kappa^2 [\tilde{\eta}^2(\kappa^2 - 2\omega_1^2) - \omega_1^2] + 2C_2\beta\tilde{\eta}^2\kappa^2\omega_2 \\ &\quad + \alpha^2\omega_1^2[(2+3\tilde{\eta}^2)\omega_1^2 - \omega_2^2 - (1+2\tilde{\eta}^2)\kappa^2] \\ &\quad + C_2^2\tilde{\eta}^2(\omega_2^2 - (2+3\tilde{\eta}^2)\omega_1^2) \}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} &\chi_m \chi_n \chi_p + \chi_m \chi_n \chi_\eta + \chi_m \chi_p \chi_\eta + \chi_n \chi_p \chi_\eta \\ &= -\frac{1+\tilde{\eta}^2}{\tilde{\eta}^2 \alpha^2 \omega_1^4} [C_2^2(1+3\tilde{\eta}^2)\omega_1^2 - 2C_2\beta\kappa^2\omega_2 - C_2^2\omega_2^2 \\ &\quad - (\kappa^2 - \omega_1^2)(\beta^2\kappa^2 - \alpha^2\omega_1^2)], \end{aligned} \quad (3.11)$$

$$\chi_m \chi_n \chi_p \chi_\eta = -\frac{C_2^2(1+\tilde{\eta}^2)^2}{\tilde{\eta}^2 \alpha^2 \omega_1^2}. \quad (3.12)$$

The solution $\xi(\chi)$ of (3.8) is

$$\begin{aligned} \xi(\chi) &= \frac{\alpha^2 - \beta^2}{\tilde{\eta}\alpha\omega_1\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \\ &\quad \times \mathbf{F}\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, \right. \\ &\quad \left. \frac{(\chi_p - \chi_m)(\chi_\eta - \chi_n)}{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}\right), \end{aligned} \quad (3.13)$$

where \mathbf{F} is the incomplete elliptic integral of first kind and

$$\chi_\eta > \chi_p > \chi > \chi_m > \chi_n.$$

Inverting $\xi(\chi)$ to $\chi(\xi)$, one finds

$$\chi(\xi) = \frac{\chi_\eta(\chi_p - \chi_n)\mathbf{DN}^2(x, m) + (\chi_\eta - \chi_p)\chi_n}{(\chi_p - \chi_n)\mathbf{DN}^2(x, m) + \chi_\eta - \chi_p}, \quad (3.14)$$

where $\mathbf{DN}(x, m)$ is one of the Jacobi elliptic functions and

$$x = \frac{\tilde{\eta}\alpha\omega_1\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}}{\alpha^2 - \beta^2} \xi,$$

$$m = \frac{(\chi_p - \chi_m)(\chi_\eta - \chi_n)}{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}.$$

By using (3.8) we can find the explicit solutions for the isometric angles ϕ_1, ϕ_2 . They are given by

$$\begin{aligned} \phi_1(\tau, \sigma) &= \omega_1\tau + \frac{1}{\tilde{\eta}\alpha\omega_1^2(\chi_\eta - 1)\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \\ &\quad \times \left\{ [\beta(\kappa^2 + \omega_1^2(\chi_\eta - 1) + C_2\omega_2)] \right. \\ &\quad \times \mathbf{F}\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, m\right) \\ &\quad - \frac{(\chi_\eta - \chi_p)(\beta\kappa^2 + C_2\omega_2)}{1 - \chi_p} \\ &\quad \left. \times \mathbf{\Pi}\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, \right. \right. \\ &\quad \left. \left. -\frac{(\chi_\eta - 1)(\chi_p - \chi_m)}{(1 - \chi_p)(\chi_\eta - \chi_m)}, m\right)\right\}, \end{aligned} \quad (3.15)$$

where $\mathbf{\Pi}$ is the incomplete elliptic integral of third kind.

$$\begin{aligned} \phi_2(\tau, \sigma) &= \omega_2\tau + \frac{1}{\tilde{\eta}\alpha\omega_1\chi_\eta\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \\ &\quad \times \left\{ [C_2(1 - \tilde{\eta}^2(\chi_\eta - 1)) + \beta\omega_2\chi_\eta] \right. \\ &\quad \times \mathbf{F}\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, m\right) \\ &\quad + \frac{C_2(1 + \tilde{\eta}^2)(\chi_\eta - \chi_p)}{\chi_p} \\ &\quad \left. \times \mathbf{\Pi}\left(\arcsin\sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, \right. \right. \\ &\quad \left. \left. \frac{\chi_\eta(\chi_p - \chi_m)}{(\chi_\eta - \chi_m)\chi_p}, m\right)\right\}, \end{aligned} \quad (3.16)$$

Now, let us go to the computations of the conserved charges Q_μ , i.e. the string energy E_s and the two angular momenta J_1, J_2 . Starting with

$$Q_\mu = \int d\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)}, \quad X^\mu = (t, \phi_1, \phi_2),$$

and applying the ansatz (3.3), one finds

$$E_s = \frac{T}{\tilde{\eta}} \left(1 - \frac{\beta^2}{\alpha^2}\right) \frac{\kappa}{\omega_1} \times \int_{\chi_m}^{\chi_p} \frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}}, \quad (3.17)$$

$$\begin{aligned} J_1 &= \frac{T}{\tilde{\eta}} \left[\left(1 - \frac{\beta(\beta\kappa^2 + C_2\omega_2)}{\alpha^2\omega_1^2}\right) \right. \\ &\quad \times \int_{\chi_m}^{\chi_p} \frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} \\ &\quad \left. - \int_{\chi_m}^{\chi_p} \frac{\chi d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} \right], \end{aligned} \quad (3.18)$$

$$\begin{aligned}
 J_2 = & \frac{T}{\tilde{\eta}^3} \left[\left(1 + \frac{1}{\tilde{\eta}^2} \right) \frac{\omega_2}{\omega_1} \right. \\
 & \times \int_{\chi_m}^{\chi_p} \frac{d\chi}{\left(1 + \frac{1}{\tilde{\eta}^2} - \chi \right) \sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} \\
 & - \left(\frac{\omega_2}{\omega_1} - \tilde{\eta}^2 \frac{\beta C_2}{\alpha^2 \omega_1} \right) \\
 & \left. \times \int_{\chi_m}^{\chi_p} \frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} \right]. \quad (3.19)
 \end{aligned}$$

We will need also the expression for the angular difference $\Delta\phi_1$. The computations give the following result

$$\begin{aligned}
 \Delta\phi_1 = & \frac{1}{\tilde{\eta}} \left[\frac{\beta}{\alpha} \int_{\chi_m}^{\chi_p} \frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} \right. \\
 & - \left(\frac{\beta \kappa^2}{\alpha \omega_1^2} + \frac{\omega_2 C_2}{\alpha \omega_1^2} \right) \\
 & \left. \times \int_{\chi_m}^{\chi_p} \frac{d\chi}{(1 - \chi) \sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n)}} \right]. \quad (3.20)
 \end{aligned}$$

Solving the integrals in (3.17)–(3.20) and introducing the notations

$$\begin{aligned}
 v = -\frac{\beta}{\alpha}, \quad u = \frac{\omega_2}{\omega_1}, \quad W = \frac{\kappa^2}{\omega_1^2}, \quad K_2 = \frac{C_2}{\alpha \omega_1}, \\
 \epsilon = \frac{(\chi_\eta - \chi_p)(\chi_m - \chi_n)}{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}, \quad (3.21)
 \end{aligned}$$

we finally obtain

$$E_s = \frac{2T}{\tilde{\eta}} \frac{(1 - v^2)\sqrt{W}}{\sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \mathbf{K}(1 - \epsilon), \quad (3.22)$$

$$\begin{aligned}
 J_1 = & \frac{2T}{\tilde{\eta} \sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \left[(1 - v^2 W + K_2 u v - \chi_\eta) \right. \\
 & \left. \times \mathbf{K}(1 - \epsilon) + (\chi_\eta - \chi_p) \mathbf{\Pi} \left(\frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, 1 - \epsilon \right) \right], \quad (3.23)
 \end{aligned}$$

$$\begin{aligned}
 J_2 = & \frac{2T}{\tilde{\eta}^3 \sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \\
 & \times \left\{ \frac{(1 + \frac{1}{\tilde{\eta}^2})u}{(1 + \frac{1}{\tilde{\eta}^2} - \chi_\eta)} \left[\mathbf{K}(1 - \epsilon) - \frac{\chi_\eta - \chi_p}{1 + \frac{1}{\tilde{\eta}^2} - \chi_p} \right. \right. \\
 & \left. \left. \times \mathbf{\Pi} \left(\frac{(\chi_p - \chi_m)(1 + \frac{1}{\tilde{\eta}^2} - \chi_\eta)}{(\chi_\eta - \chi_m)(1 + \frac{1}{\tilde{\eta}^2} - \chi_p)}, 1 - \epsilon \right) \right] \right. \\
 & \left. - (u + \tilde{\eta}^2 K_2 v) \mathbf{K}(1 - \epsilon) \right\}, \quad (3.24)
 \end{aligned}$$

$$\begin{aligned}
 \Delta\phi_1 = & \frac{1}{\tilde{\eta} \sqrt{(\chi_\eta - \chi_m)(\chi_p - \chi_n)}} \\
 & \times \left\{ \frac{vW - K_2 u}{(\chi_\eta - 1)(1 - \chi_p)} \left[(\chi_\eta - \chi_p) \right. \right. \\
 & \left. \left. \times \mathbf{\Pi} \left(-\frac{(\chi_\eta - 1)(\chi_p - \chi_m)}{(\chi_\eta - \chi_m)(1 - \chi_p)}, 1 - \epsilon \right) \right. \right. \\
 & \left. \left. - (1 - \chi_p) \mathbf{K}(1 - \epsilon) \right] - v \mathbf{K}(1 - \epsilon) \right\}, \quad (3.25)
 \end{aligned}$$

where \mathbf{K} and $\mathbf{\Pi}$ are the complete elliptic integrals of first and third kind.

4. Small ϵ -expansions and dispersion relation

In this section we restrict ourselves to the simpler case of giant magnons with one nonzero angular momentum. To this end, we set the second isometric angle $\phi_2 = 0$. From the solution (3.16) it is clear that ϕ_2 is zero when

$$\begin{aligned}
 \omega_2 = C_2 = 0, \\
 \text{or equivalently (see (3.21))}
 \end{aligned}$$

$$u = K_2 = 0.$$

Then it follows from (3.12) that $\chi_n = 0$ because $\chi_\eta > \chi_p > \chi_m > 0$ for the finite-size case. In addition, we express χ_m through the other parameters in correspondence with (3.21)

$$\chi_m = \frac{\chi_\eta \chi_p}{\chi_\eta - (1 - \epsilon)\chi_p} \epsilon.$$

As a consequence (3.9)–(3.11) take the form

$$\frac{(1 - \epsilon)\chi_p^2 - 2\epsilon\chi_p\chi_\eta - \chi_\eta^2}{\chi_\eta - (1 - \epsilon)\chi_p} + 3 - (1 + v^2)W + \frac{1}{\tilde{\eta}^2} = 0, \quad (4.1)$$

$$\begin{aligned}
 \chi_p\chi_\eta + \frac{\epsilon\chi_p\chi_\eta(\chi_p + \chi_\eta)}{\chi_\eta - (1 - \epsilon)\chi_p} \\
 - \frac{2 - (1 + v^2)W + (3 - (2 + v^2(2 - W))W)\tilde{\eta}^2}{\tilde{\eta}^2} = 0, \quad (4.2)
 \end{aligned}$$

$$\frac{\epsilon\chi_p^2\chi_\eta^2}{\chi_\eta - (1 - \epsilon)\chi_p} - \frac{(1 + \tilde{\eta}^2)(1 - W)(1 - v^2W)}{\tilde{\eta}^2} = 0. \quad (4.3)$$

In order to obtain the leading finite-size effect on the dispersion relation, we consider the limit $\epsilon \rightarrow 0$ in (4.1)–(4.3) first. We will use the following small ϵ -expansions for the remaining parameters

$$\begin{aligned}
 \chi_\eta = \chi_{\eta 0} + (\chi_{\eta 1} + \chi_{\eta 2} \log \epsilon)\epsilon \\
 \chi_p = \chi_{p 0} + (\chi_{p 1} + \chi_{p 2} \log \epsilon)\epsilon, \\
 v = v_0 + (v_1 + v_2 \log \epsilon)\epsilon, \\
 W = 1 + W_1 \epsilon. \quad (4.4)
 \end{aligned}$$

Replacing (4.4) into (4.1)–(4.3) and expanding in ϵ one finds the following solution of the resulting equations

$$\begin{aligned}
 \chi_{p 0} = 1 - v_0^2, \quad \chi_{p 1} = 1 - v_0^2 - 2v_0 v_1 - \frac{(1 - v_0^2)^2}{1 + \tilde{\eta}^2 v_0^2}, \\
 \chi_{p 2} = -2v_0 v_2, \\
 \chi_{\eta 0} = 1 + \frac{1}{\tilde{\eta}^2}, \quad \chi_{\eta 1} = \chi_{\eta 2} = 0, \\
 W_1 = -\frac{(1 + \tilde{\eta}^2)(1 - v_0^2)}{1 + \tilde{\eta}^2 v_0^2}. \quad (4.5)
 \end{aligned}$$

Next, we expand $\Delta\phi_1$ in ϵ and impose the condition that the resulting expression does not depend on ϵ . After using (4.5) this gives

$$\Delta\phi_1 = 2 \operatorname{arccot} \left(v_0 \sqrt{\frac{1 + \tilde{\eta}^2}{1 - v_0^2}} \right) \quad (4.6)$$

and two equations with solution

$$v_1 = \frac{v_0(1 - v_0^2)[1 - \log 16 + \tilde{\eta}^2(2 - v_0^2(1 + \log 16))]}{4(1 + \tilde{\eta}^2 v_0^2)},$$

$$v_2 = \frac{1}{4}v_0(1 - v_0^2). \quad (4.7)$$

Solving (4.6) with respect to v_0 one finds

$$v_0 = \frac{\cot \frac{\Delta\phi_1}{2}}{\sqrt{\tilde{\eta}^2 + \csc^2 \frac{\Delta\phi_1}{2}}}. \quad (4.8)$$

Now let us go to the ϵ -expansion of the difference $E_s - J_1$. Taking into account the solutions for the parameters, it can be written as

$$E_s - J_1 = 2g\sqrt{1 + \tilde{\eta}^2} \left[\frac{1}{\tilde{\eta}} \operatorname{arcsinh} \left(\tilde{\eta} \sin \frac{p}{2} \right) - \frac{(1 + \tilde{\eta}^2) \sin^3 \frac{p}{2}}{4\sqrt{1 + \tilde{\eta}^2} \sin^2 \frac{p}{2}} \epsilon \right], \quad (4.9)$$

where the expression for ϵ can be found from the expansion of J_1 . To the leading order, the result is

$$\epsilon = 16 \exp \left[- \left(\frac{J_1}{g} + \frac{2\sqrt{1 + \tilde{\eta}^2}}{\tilde{\eta}} \operatorname{arcsinh} \left(\tilde{\eta} \sin \frac{p}{2} \right) \right) \times \sqrt{\frac{1 + \tilde{\eta}^2 \sin^2 \frac{p}{2}}{(1 + \tilde{\eta}^2) \sin^2 \frac{p}{2}}} \right]. \quad (4.10)$$

In writing (4.9), (4.10), we used (2.6) and identified the angular difference $\Delta\phi_1$ with the magnon momentum p in the dual spin chain.

For $\epsilon = 0$, (4.9) reduces to the dispersion relation for the infinite-size giant magnon obtained in [7] for the large g case. In the limit $\tilde{\eta} \rightarrow 0$, (4.9) gives the correct result for the undeformed case found in [10].

5. Concluding remarks

Here we dealt with strings moving in the $R_t \times S_\eta^3$ subspace of the η -deformed $AdS_5 \times S^5$. The finite-size dyonic giant magnon solution is contained in the string configurations we considered.

We derived the explicit exact solutions for the string coordinates and the corresponding conserved charges. Then we restricted ourselves to the case of giant magnons with one nonzero angular momentum and obtained the dispersion relation for them including the leading finite-size effect on it.

It will be interesting to extend the result (4.9), (4.10) to the case of dyonic giant magnons. We will report on this soon.

Another possible direction of further investigation is to show that (4.9), (4.10) can be reproduced by using the Lüscher formula for the finite-size effects on the dispersion relation and we are going to do that in the near future.

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