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String solutions in $\mathrm{AdS}_3 \times S^3 \times T^4$ with NS-NS B-field

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We develop an approach for solving the string equations of motion and Virasoro constraints in any background that has some (unfixed) number of commuting Killing vector fields. It is based on a specific ansatz for the string embedding. We apply the above-mentioned approach for strings moving in $\mathrm{AdS}_3 \times S^3 \times T^4$ with a 2-form NS-NS B-field. We succeeded to find solutions for a large class of string configurations on this background. In particular, we derive dyonic giant magnon solutions in the $R_t \times S^3$ subspace and obtain the leading finite-size correction to the dispersion relation.

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I. INTRODUCTION

A very important development in the field of string theory has been achieved for the case of AdS/CFT duality [1] between strings and conformal field theories in various dimensions. The most developed case is the correspondence between strings living in $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ SYM in four dimensions. Another example is the duality between strings on a $\mathrm{AdS}_4 \times CP^3$ background and $\mathcal{N} = 6$ super Chern-Simons-matter theory in three space-time dimensions. The main achievements in the above examples are due to the discovery of integrable structures on both sides of the correspondence. Many other cases have been considered also [2].

Typically, classical string solutions provide dual CFT states whose conformal dimensions are identified with energies of the string configurations. (See a review by [3].) One of the frequently studied string states in AdS/CFT duality is a giant magnon that lives on a S^2 subspace of the string target space [4]. This state is important because it corresponds to fundamental excitations on the world sheet whose S matrix is at the core of the nonperturbative integrability.

For the case of AdS_3/CFT_2 duality [5–26], a large coupling constant (large string tension) limit plays a particularly important role since there are still not much understanding on the CFT side. (For a recent revew, see [27].) One interesting feature of this duality is the existence of nontrivial B fields in the string action due to a NS-NS flux and its effect on the classical string solutions. Here, we will focus on the giant magnon solution for the relatively simpler background geometry of $\mathrm{AdS}_3 \times S^3 \times T^4$ with the NS-NS B fields. Our main result is to compute the finite-size correction for the giant magnon dispersion relation which can provide a stringent test for the world-sheet S matrix.

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The paper is organized as follows. In Sec. II we present our general approach to string dynamics in curved backgrounds with the B-fields. In Sec. III we apply it to strings moving in $\mathrm{AdS}_3 \times S^3 \times T^4$ with the NS-NS B-field background. In Sec. IV we restrict ourselves to a giant magnon solution and derive the dispersion relation, including the leading finite-size effect on it. Section V is devoted to our concluding remarks.

II. STRINGS IN CURVED BACKGROUNDS WITH A B-FIELD: THE GENERAL APPROACH

Considering string dynamics in curved backgrounds with a B-field, we develop an approach that will allow us to obtain exact string solutions in sufficiently general string theory target spaces.

A. Bosonic string action, equations of motion and constraints

In our further considerations, we will use the Polyakov type action for the bosonic string in a D-dimensional curved space-time with metric tensor $g_{MN}(x)$, interacting with a background 2-form gauge field $b_{MN}(x)$ via a Wess-Zumino term,

$$S^P = \int d^2\xi \mathcal{L}^P,$$

$$\mathcal{L}^P = -\frac{1}{2}(T\sqrt{-\gamma}\gamma^{mn}G_{mn} - Q\epsilon^{mn}B_{mn}),$$

$$\xi^m = (\xi^0, \xi^1) = (\tau, \sigma), m, n = 0, 1,$$

where

$$G_{mn} = \partial_m X^M \partial_n X^N g_{MN},$$

$$B_{mn} = \partial_m X^M \partial_n X^N b_{MN},$$

$$(\partial_m = \partial/\partial\xi^m, \quad M, N = 0, 1, \ldots, D-1)$$

are the fields induced on the string world sheet, γ is the determinant of the auxiliary world-sheet metric γ_{mn}, and γ^{mn} is its inverse. The position of the string in the background space-time is given by $x^M = X^M(\xi^m)$; $T = 1/2\pi\alpha'$ and Q are the string tension and charge respectively. If we consider the action S^P as a bosonic part of a supersymmetric one, we have to put $Q = \pm T$. In what follows, $Q = T$.

The equations of motion for X^M following from S^P are

$$-g_{LK}[\partial_m(\sqrt{-\gamma}\gamma^{mn}\partial_n X^K) + \sqrt{-\gamma}\gamma^{mn}\Gamma^K_{MN}\partial_m X^M \partial_n X^N] = \frac{1}{2}H_{LMN}\epsilon^{mn}\partial_m X^M \partial_n X^N, \tag{2.1}$$

where $(\partial_M = \partial/\partial x^M)$,

$$\Gamma_{L,MN} = g_{LK}\Gamma^K_{MN} = \frac{1}{2}(\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN}),$$
$$H_{LMN} = \partial_L b_{MN} + \partial_M b_{NL} + \partial_N b_{LM}$$

are the components of the symmetric connection corresponding to the metric g_{MN} and the field strength of the gauge field b_{MN}, respectively. The constraints are obtained by varying the action S^P with respect to γ_{mn}:

$$\delta_{\gamma_{mn}} S^P = 0 \Rightarrow (\gamma^{kl}\gamma^{mn} - 2\gamma^{km}\gamma^{ln})G_{mn} = 0. \tag{2.2}$$

B. Gauge choice and ansatz

In what follow we will use conformal gauge $\gamma^{mn} = \eta^{mn} = \mathrm{diag}(-1,1)$ in which the string Lagrangian, the Virasoro constraints and the equations of motion take the following form:

$$\mathcal{L} = \frac{T}{2}(G_{00} - G_{11} + 2B_{01}),$$
$$G_{00} + G_{11} = 0, \qquad G_{01} = 0,$$
$$g_{LK}[(\partial_0^2 - \partial_1^2)X^K + \Gamma^K_{MN}(\partial_0 X^M \partial_0 X^N - \partial_1 X^M \partial_1 X^N)] = H_{LMN}\partial_0 X^M \partial_1 X^N. \tag{2.3}$$

Now, let us *suppose* that there exists some number of commuting Killing vector fields along part of the X^M coordinates and split X^M into two parts,

$$X^M = (X^\mu, X^a),$$

where X^μ are the isometric coordinates and X^a are the nonisometric ones. The existence of isometric coordinates leads to the following conditions on the background fields:

$$\partial_\mu g_{MN} = 0, \qquad \partial_\mu b_{MN} = 0. \tag{2.4}$$

Then from the string action, we can compute the conserved charges,

$$Q_\mu = \int d\sigma \frac{\partial\mathcal{L}}{\partial(\partial_0 X^\mu)} \tag{2.5}$$

under the above conditions.

Next, we introduce the following ansatz for the string embedding

$$X^\mu(\tau,\sigma) = \Lambda^\mu\tau + \tilde{X}^\mu(\alpha\sigma + \beta\tau),$$
$$X^a(\tau,\sigma) = \tilde{X}^a(\alpha\sigma + \beta\tau), \tag{2.6}$$

where Λ^μ, α, β are arbitrary parameters. Further on, we will use the notation $\xi = \alpha\sigma + \beta\tau$. Applying this ansatz, one can find that the equalities (2.3), (2.5) become

$$\mathcal{L} = \frac{T}{2}\left[-(\alpha^2 - \beta^2)g_{MN}\frac{d\tilde{X}^M}{d\xi}\frac{d\tilde{X}^N}{d\xi} + 2\Lambda^\mu(\beta g_{\mu N} + \alpha b_{\mu N})\frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu\Lambda^\nu g_{\mu\nu}\right], \tag{2.7}$$

$$G_{00} + G_{11} = (\alpha^2 + \beta^2)g_{MN}\frac{d\tilde{X}^M}{d\xi}\frac{d\tilde{X}^N}{d\xi} + 2\beta\Lambda^\mu g_{\mu N}\frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu\Lambda^\nu g_{\mu\nu} = 0, \tag{2.8}$$

$$G_{01} = \alpha\beta g_{MN}\frac{d\tilde{X}^M}{d\xi}\frac{d\tilde{X}^N}{d\xi} + \alpha\Lambda^\mu g_{\mu N}\frac{d\tilde{X}^N}{d\xi} = 0, \tag{2.9}$$

$$-(\alpha^2 - \beta^2)\left[g_{LK}\frac{d^2\tilde{X}^K}{d\xi^2} + \Gamma_{L,MN}\frac{d\tilde{X}^M}{d\xi}\frac{d\tilde{X}^N}{d\xi}\right] + 2\beta\Lambda^\mu\Gamma_{L,\mu N}\frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu\Lambda^\nu\Gamma_{L,\mu\nu} = \alpha\Lambda^\mu H_{L\mu N}\frac{d\tilde{X}^N}{d\xi}, \tag{2.10}$$

$$Q_\mu = \frac{T}{\alpha}\int d\xi\left[(\beta g_{\mu N} + \alpha b_{\mu N})\frac{d\tilde{X}^N}{d\xi} + \Lambda^\nu g_{\mu\nu}\right]. \tag{2.11}$$

Our next task is to try to solve the equations of motion (2.10) for the isometric coordinates, i.e. for $L = \lambda$. Due to the conditions (2.4) imposed on the background fields, we obtain that

$$\Gamma_{\lambda,ab} = \frac{1}{2}(\partial_a g_{b\lambda} + \partial_b g_{a\lambda}), \qquad \Gamma_{\lambda,\mu a} = \frac{1}{2}\partial_a g_{\mu\lambda} \;\; \Gamma_{\lambda,\mu\nu} = 0,$$
$$H_{\lambda ab} = \partial_a b_{b\lambda} + \partial_b b_{\lambda a}, \qquad H_{\lambda\mu a} = \partial_a b_{\lambda\mu}, \qquad H_{\lambda\mu\nu} = 0.$$

By using this, one can find the following first integrals for $\tilde{X}^\mu$,

$$\frac{d\tilde{X}^\mu}{d\xi} = \frac{1}{\alpha^2 - \beta^2}[g^{\mu\nu}(C_\nu - \alpha\Lambda^\rho b_{\nu\rho}) + \beta\Lambda^\mu] - g^{\mu\nu} g_{\nu a}\frac{d\tilde{X}^a}{d\xi}, \tag{2.12}$$

where C_ν are arbitrary integration constants. Therefore, according to our ansatz (2.6), the solutions for the string coordinates X^μ can be written as

$$X^\mu(\tau,\sigma) = \Lambda^\mu\tau + \frac{1}{\alpha^2 - \beta^2}\int d\xi[g^{\mu\nu}(C_\nu - \alpha\Lambda^\rho b_{\nu\rho}) + \beta\Lambda^\mu] - \int g^{\mu\nu} g_{\nu a} d\tilde{X}^a(\xi). \tag{2.13}$$

Now, let us turn to the remaining equations of motion corresponding to $L = a$, where

$$\Gamma_{a,\mu b} = -\frac{1}{2}(\partial_a g_{b\mu} - \partial_b g_{a\mu}), \qquad \Gamma_{a,\mu\nu} = -\frac{1}{2}\partial_a g_{\mu\nu}, \qquad H_{a\mu\nu} = \partial_a b_{\mu\nu}, \qquad H_{a\mu b} = -\partial_a b_{b\mu} + \partial_b b_{a\mu}.$$

Taking this into account and replacing the first integrals for $\tilde{X}^\mu$ already found, one can write these equations in the form (prime is used for $d/d\xi$)

$$(\alpha^2 - \beta^2)[h_{ab}\tilde{X}^{b''} + \Gamma^h_{a,bc}\tilde{X}^{b'}\tilde{X}^{c'}] = 2\partial_{[a}A_{b]}\tilde{X}^{b'} - \partial_a U, \tag{2.14}$$

where

$$h_{ab} = g_{ab} - g_{a\mu}g^{\mu\nu}g_{\nu b}, \qquad \Gamma^h_{a,bc} = \frac{1}{2}(\partial_b h_{ca} + \partial_c h_{ba} - \partial_a h_{bc}) \tag{2.15}$$

$$A_a = g_{a\mu}g^{\mu\nu}(C_\nu - \alpha\Lambda^\rho b_{\nu\rho}) + \alpha\Lambda^\mu b_{a\mu}, \tag{2.16}$$

$$U = \frac{1/2}{\alpha^2 - \beta^2}[(C_\mu - \alpha\Lambda^\rho b_{\mu\rho})g^{\mu\nu}(C_\nu - \alpha\Lambda^\lambda b_{\nu\lambda}) + \alpha^2\Lambda^\mu\Lambda^\nu g_{\mu\nu}]. \tag{2.17}$$

One can show that the above equations for $\tilde{X}^a$ can be derived from the effective Lagrangian,

$$\mathcal{L}^{\rm eff}(\xi) = \frac{1}{2}(\alpha^2 - \beta^2)h_{ab}\tilde{X}^{a'}\tilde{X}^{b'} + A_a\tilde{X}^{a'} - U.$$

The corresponding effective Hamiltonian is

$$\mathcal{H}^{\rm eff}(\xi) = \frac{1}{2}(\alpha^2 - \beta^2)h_{ab}\tilde{X}^{a'}\tilde{X}^{b'} + U,$$

or in terms of the momenta p_a conjugated to $\tilde{X}^a$,

$$\mathcal{H}^{\rm eff}(\xi) = \frac{1}{2}(\alpha^2 - \beta^2)h^{ab}(p_a - A_a)(p_b - A_b) + U.$$

The Virasoro constraints (2.8), (2.9) become

$$\frac{1}{2}(\alpha^2 - \beta^2)h_{ab}\tilde{X}^{a'}\tilde{X}^{b'} + U = 0, \qquad \alpha\Lambda^\mu C_\mu = 0. \tag{2.18}$$

Finally, let us write down the expressions for the conserved charges (2.11),

$$Q_\mu = \frac{T}{\alpha^2-\beta^2}\int d\xi \left[\frac{\beta}{\alpha}C_\mu + \alpha\Lambda^\nu g_{\mu\nu} + b_{\mu\nu}g^{\nu\rho}(C_\rho - \alpha\Lambda^\lambda b_{\rho\lambda}) + (\alpha^2-\beta^2)(b_{\mu a} - b_{\mu\nu}g^{\nu\rho}g_{\rho a})\tilde{X}^{a'}\right]. \tag{2.19}$$

III. STRINGS IN $\mathrm{AdS}_3 \times S^3 \times T^4$ WITH NS-NS B-FIELD

The background geometry of this target space can be written in the following form[1]:

$$\begin{aligned} ds^2_{\mathrm{AdS}_3} &= -(1+r^2)dt^2 + (1+r^2)^{-1}dr^2 + r^2 d\phi^2, \qquad b_{t\phi} = qr^2, \\ ds^2_{S^3} &= d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2, \qquad b_{\phi_1\phi_2} = -q\cos^2\theta, \\ ds^2_{T^4} &= (d\varphi^i)^2, \qquad i = 1,2,3,4. \end{aligned}$$

Now, we apply the formulation given in Sec. II. According to our notations,

$$\begin{aligned} &X^\mu = (t,\phi,\phi_1,\phi_2,\varphi^i), \qquad X^a = (r,\theta), \\ &g_{\mu\nu} = (g_{tt}, g_{\phi\phi}, g_{\phi_1\phi_1}, g_{\phi_2\phi_2}, g_{ij}), \qquad g_{ab} = (g_{rr}, g_{\theta\theta}), \qquad g_{a\mu} = 0, \qquad h_{ab} = g_{ab}, \\ &b_{\mu\nu} = (b_{t\phi}, b_{\phi_1\phi_2}), \qquad b_{a\nu} = 0, \\ &A_a = 0, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} g_{tt} &= (g^{tt})^{-1} = -(1+r^2), \qquad g_{\phi\phi} = (g^{\phi\phi})^{-1} = r^2, \qquad g_{\phi_1\phi_1} = (g^{\phi_1\phi_1})^{-1} = \sin^2\theta, \\ g_{\phi_2\phi_2} &= (g^{\phi_2\phi_2})^{-1} = \cos^2\theta, \qquad g_{ij} = (g^{ij})^{-1} = \delta_{ij}, \\ g_{rr} &= (g^{rr})^{-1} = (1+r^2)^{-1}, \qquad g_{\theta\theta} = 1, \\ b_{t\phi} &= qr^2, \qquad b_{\phi_1\phi_2} = -q\cos^2\theta. \end{aligned} \tag{3.2}$$

Since $g_{a\mu} = 0$, the solutions (2.13) for the coordinates X^μ are simplified to

$$X^\mu(\tau,\sigma) = \Lambda^\mu\tau + \tilde{X}^\mu(\xi) = \Lambda^\mu\tau + \frac{1}{\alpha^2-\beta^2}\int d\xi[g^{\mu\nu}(C_\nu - \alpha\Lambda^\rho b_{\nu\rho}) + \beta\Lambda^\mu], \tag{3.3}$$

where $g^{\mu\nu}$ and $b_{\nu\rho}$ must be replaced from above.

Now, we want to find the solutions for the nonisometric string coordinates X^a. To this end we have to solve Eqs. (2.14), which in the case at hand reduce to

$$(\alpha^2-\beta^2)[g_{ab}\tilde{X}^{b''} + \Gamma_{a,bc}\tilde{X}^{b'}\tilde{X}^{c'}] + \partial_a\sum_{b=r,\theta}U_b = 0, \tag{3.4}$$

where the scalar potential U in (2.17) is represented as a sum of two parts: $U_r = U_r(r)$ for the AdS_3 subspace and $U_\theta = U_\theta(\theta)$ for the S^3 subspace of the background.

Taking into account that the metric g_{ab} is diagonal, one can find the following two first integrals of (3.4):

$$\tilde{X}^{a'} = \sqrt{\frac{C_a - 2U_a}{(\alpha^2-\beta^2)g_{aa}}}. \tag{3.5}$$

It follows from here that

[1]The common radius R of the three subspaces is set to 1, and q is the parameter used in [25].

$$d\xi = \frac{d\tilde{X}^a}{\sqrt{\frac{C_a - 2U_a}{(\alpha^2-\beta^2)g_{aa}}}}. \tag{3.6}$$

So, we have two different expressions for $d\xi$, which obviously must coincide. This is a condition for self-consistency. It leads to

$$\int \frac{dr}{\sqrt{\frac{C_r-2U_r}{g_{rr}}}} = \int \frac{d\theta}{\sqrt{\frac{C_\theta-2U_\theta}{g_{\theta\theta}}}}, \tag{3.7}$$

which actually gives implicitly the "orbit" $r(\theta)$, i.e. how the radial coordinate r on AdS_3 depends on the angle θ in S^3.

Now, we have to check if the first integrals for $\tilde{X}^a(\xi)$ are compatible with the Virasoro constraints (2.18). Replacing $\tilde{X}^{a'}$ in the first of them, one finds

$$C_r + C_\theta = 0.$$

Thus, we found all first integrals of the string equations of motion, compatible with the Virasoro constraints, which reduce to algebraic relations between the embedding parameters and the integration constants.

Now, let us give the expressions for the conserved charges (2.19), corresponding to the isometric coordinates,

$$-Q_t \equiv E_s = \frac{T}{\alpha^2-\beta^2}\left[\left(\alpha\Lambda^t - \frac{\beta}{\alpha}C_t - qC_\phi\right)\int d\xi + \alpha(1-q^2)\Lambda^t \int d\xi r^2\right], \tag{3.8}$$

$$Q_\phi \equiv S = \frac{T}{\alpha^2-\beta^2}\left[\left(\frac{\beta}{\alpha}C_\phi + qC_t + q^2\alpha\Lambda^\phi\right)\int d\xi + (1-q^2)\alpha\Lambda^\phi \int d\xi r^2 - (qC_t + q^2\alpha\Lambda^\phi)\int \frac{d\xi}{1+r^2}\right], \tag{3.9}$$

$$\begin{aligned}
Q_{\phi_1} \equiv J_1 &= \frac{T}{\alpha^2-\beta^2}\left[\left(\frac{\beta}{\alpha}C_{\phi_1} + \alpha\Lambda^{\phi_1} - qC_{\phi_2}\right)\int d\xi - (1-q^2)\alpha\Lambda^{\phi_1}\int \cos^2\theta d\xi\right],\\
Q_{\phi_2} \equiv J_2 &= \frac{T}{\alpha^2-\beta^2}\left[\left(\frac{\beta}{\alpha}C_{\phi_2} - q(C_{\phi_1} + q\alpha\Lambda^{\phi_2})\right)\int d\xi + (1-q^2)\alpha\Lambda^{\phi_2}\int \cos^2\theta d\xi + q(C_{\phi_1} + q\alpha\Lambda^{\phi_2})\int \frac{d\xi}{1-\cos^2\theta}\right],
\end{aligned} \tag{3.10}$$

$$Q_i \equiv J_i^T = \frac{T}{\alpha^2-\beta^2}\left(\frac{\beta}{\alpha}C_i + \alpha\Lambda^j\delta_{ij}\right)\int d\xi. \tag{3.11}$$

Here we used the following notations: E_s is the string energy, S is the spin in AdS_3, J_1 and J_2 are the two angular momenta in S^3, while J_i^T are the four angular momenta on T^4.

The explicit expressions for the string coordinates, the "orbit" $r(\theta)$, and the conserved charges in this background are given in the Appendix.

IV. GIANT MAGNON SOLUTION

The giant magnon string solution was found in [4]. It is a specific string configuration, living in the $R_t \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$ with an angular momentum J_1 which goes to ∞. A similar configuration, the dyonic giant magnon, has been obtained in [28] which moves in $R_t \times S^3$ subspace with two angular momenta J_1, J_2 with $J_1 \to \infty$. These classical configurations have played an important role in understanding exact, quantum aspects of the AdS/CFT correspondence. In particular, corrections due to a large but finite J_1 obtained in [29] and [30] can provide a nontrivial check for the exact world-sheet S matrix.

In this section we provide similar string solutions in $\mathrm{AdS}_3 \times S^3 \times T^4$ with NS-NS B-field for a large but finite J_1. A giant magnon solution with infinite angular momentum has been constructed in a recent paper [25] with a dispersion relation,[2]

[2]The terms proportional to q are due to the nonzero B-field on S^3.

$$E_s - J_1 = \sqrt{(J_2 - qT\Delta\phi_1)^2 + 4T^2(1-q^2)\sin^2\frac{\Delta\phi_1}{2}}. \tag{4.1}$$

This relation is already quite different from those for the ordinary (dyonic) giant magnons. We will show that there exist even bigger differences for the finite-size corrections.

A. Exact results

In order to consider dyonic giant magnon solutions, we restrict our general ansatz (2.6) in the following way:

$$\begin{aligned}
&X^t \equiv t = \kappa\tau, \quad \text{i.e.} \quad \Lambda^t = \kappa, \qquad \tilde{X}^t(\xi) = 0,\\
&X^\phi \equiv \phi = 0, \quad \text{i.e.} \quad \Lambda^\phi = 0, \qquad \tilde{X}^\phi(\xi) = 0\\
&X^r \equiv r = \tilde{X}^r(\xi) = 0,\\
&X^{\phi_1} \equiv \phi_1 = \omega_1\tau + \tilde{X}^{\phi_1}(\xi), \quad \text{i.e.} \quad \Lambda^{\phi_1} = \omega_1,\\
&X^{\phi_2} \equiv \phi_2 = \omega_2\tau + \tilde{X}^{\phi_2}(\xi), \quad \text{i.e.} \quad \Lambda^{\phi_2} = \omega_2,\\
&X^\theta \equiv \theta = \tilde{X}^\theta(\xi), \qquad X^{\varphi^i} \equiv \varphi^i = 0.
\end{aligned}$$

As a result, we can claim that

$$C_t = \beta\kappa,$$

which comes from $\frac{d\tilde{X}^t}{d\xi} = 0$.

Now, we can rewrite the first integrals for $\tilde{X}^\mu$ on S^3 as

$$\begin{aligned}
\frac{d\tilde{X}^{\phi_1}}{d\xi} &= \frac{1}{\alpha^2-\beta^2}\left[(C_{\phi_1} + q\alpha\omega_2)\frac{1}{1-\chi} + \beta\omega_1 - q\alpha\omega_2\right],\\
\frac{d\tilde{X}^{\phi_2}}{d\xi} &= \frac{1}{\alpha^2-\beta^2}\left(\frac{C_{\phi_2}}{\chi} + \beta\omega_2 - q\alpha\omega_1\right),
\end{aligned} \tag{4.2}$$

where $\chi = \cos^2\theta$.

The first Virasoro constraint, which in the case under consideration is the first integral of the equation of motion for θ, reduces to

$$\begin{aligned}
\left(\frac{d\chi}{d\xi}\right)^2 = \frac{4}{(\alpha^2-\beta^2)^2}\chi(1-\chi)\Bigg[&(\alpha^2+\beta^2)\kappa^2 - \frac{(C_{\phi_1} + q\alpha\omega_2\chi)^2}{1-\chi} - \frac{(C_{\phi_2} - q\alpha\omega_1\chi)^2}{\chi}\\
&- \alpha^2(\omega_2^2 - \omega_1^2)\chi - \alpha^2\omega_1^2\Bigg].
\end{aligned} \tag{4.3}$$

Also, the second Virasoro constraint becomes

$$\omega_1 C_{\phi_1} + \omega_2 C_{\phi_2} + \beta\kappa^2 = 0. \tag{4.4}$$

Taking (4.4) into account, we can rewrite (4.3) as

$$\left(\frac{d\chi}{d\xi}\right)^2 = 4(1-q^2)\frac{\omega_1^2}{\alpha^2}\frac{1-u^2}{(1-v^2)^2}(\chi_p - \chi)(\chi - \chi_m)(\chi - \chi_n), \tag{4.5}$$

where

$$\chi_p+\chi_m+\chi_n=\frac{-(v^2W+(W+u^2-2+q^2))+2q(uvW+K(1-u^2))}{(1-q^2)(1-u^2)},$$
$$\chi_p\chi_m+\chi_p\chi_n+\chi_m\chi_n=-\frac{(1+v^2)W+K^2-(vW-uK)^2-1+2qK}{(1-q^2)(1-u^2)},$$
$$\chi_p\chi_m\chi_n=-\frac{K^2}{(1-q^2)(1-u^2)}, \tag{4.6}$$

and we introduced the notations,

$$v=-\frac{\beta}{\alpha},\qquad u=\frac{\omega_2}{\omega_1},\qquad W=\left(\frac{\kappa}{\omega_1}\right)^2,\qquad K=\frac{C_{\phi_2}}{\alpha\omega_1}.$$

This leads to

$$d\xi=\frac{\alpha}{2\omega_1}\frac{1-v^2}{\sqrt{(1-q^2)(1-u^2)}}\frac{d\chi}{\sqrt{(\chi_p-\chi)(\chi-\chi_m)(\chi-\chi_n)}}. \tag{4.7}$$

Integrating (4.7) and inverting $\xi(\chi)$ to $\chi(\xi)\equiv\cos^2[\theta(\xi)]$, one finds the following explicit solution,

$$\chi=(\chi_p-\chi_n)\mathbf{DN}^2\left[\frac{\sqrt{(1-q^2)(1-u^2)(\chi_p-\chi_n)}}{1-v^2}\omega_1(\sigma-v\tau),\frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right]+\chi_n, \tag{4.8}$$

where **DN** is one of the Jacobi elliptic functions.

Next, we integrate (4.2), and according to our ansatz, obtain that the solutions for the isometric angles on S^3 are given by

$$\phi_1=\omega_1\tau+\frac{2}{\sqrt{(1-q^2)(1-u^2)(\chi_p-\chi_n)}}\left[\frac{vW-Ku+qu}{1-\chi_p}\Pi\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}},-\frac{\chi_p-\chi_m}{1-\chi_p},\frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right)\right.$$
$$\left.-(v+qu)\mathbf{F}\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}},\frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right)\right] \tag{4.9}$$

$$\phi_2=\omega_2\tau+\frac{2}{\sqrt{(1-q^2)(1-u^2)(\chi_p-\chi_n)}}\left[\frac{K}{\chi_p}\Pi\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}},1-\frac{\chi_m}{\chi_p},\frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right)\right.$$
$$\left.-(uv+q)\mathbf{F}\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}},\frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right)\right], \tag{4.10}$$

where **F** and Π are the incomplete elliptic integrals of the first and third kind.

By using (4.7), one can find also the conserved quantities, namely, the string energy E_s and the two angular momenta J_1, J_2,

$$E_s=2T\frac{(1-v^2)\sqrt{W}}{\sqrt{(1-q^2)(1-u^2)(\chi_p-\chi_n)}}\mathbf{K}(1-\epsilon), \tag{4.11}$$

$$J_1=\frac{2T}{\sqrt{(1-q^2)(1-u^2)(\chi_p-\chi_n)}}\{[1-v^2W+K(uv-q)]\mathbf{K}(1-\epsilon)$$
$$-(1-q^2)[\chi_n\mathbf{K}(1-\epsilon)+(\chi_p-\chi_n)\mathbf{E}(1-\epsilon)]\}, \tag{4.12}$$

$$J_2 = \frac{2T}{\sqrt{(1-q^2)(1-u^2)(\chi_p-\chi_n)}}\Big\{(1-q^2)u[\chi_n\mathbf{K}(1-\epsilon)+(\chi_p-\chi_n)\mathbf{E}(1-\epsilon)]$$
$$-[Kv+q(vW-Ku)+q^2u]\mathbf{K}(1-\epsilon)+q\frac{vW-Ku+qu}{1-\chi_p}\Pi\left(-\frac{\chi_p-\chi_n}{1-\chi_p}(1-\epsilon),1-\epsilon\right)\Big\}, \quad (4.13)$$

where $\mathbf{K}$, $\mathbf{E}$ and Π are the complete elliptic integrals of the first, second and third kind, and ϵ is defined as

$$\epsilon = \frac{\chi_m-\chi_n}{\chi_p-\chi_n}. \quad (4.14)$$

We will need also the expression for the angular difference $\Delta\phi_1$, which is found to be

$$\Delta\phi_1 = \frac{2}{\sqrt{(1-q^2)(1-u^2)(\chi_p-\chi_n)}}\left[\frac{vW-Ku+qu}{1-\chi_p}\Pi\left(-\frac{\chi_p-\chi_n}{1-\chi_p}(1-\epsilon),1-\epsilon\right)-(v+qu)\mathbf{K}(1-\epsilon)\right]. \quad (4.15)$$

The expressions (4.11), (4.12), (4.13), (4.15) are for the finite-size dyonic strings living in the $R_t\times S^3$ subspace of $AdS_3\times S^3\times T^4$.

B. Leading finite-size effect on the dispersion relation

In order to find the leading finite-size effect on the dispersion relation, we have to consider the limit $\epsilon\to 0$, since $\epsilon=0$ corresponds to the infinite-size case. In this subsection we restrict ourselves to the particular case when $\chi_n=K=0$.[3] Then the third equation in (4.6) is satisfied identically, while the other two simplify to

$$\chi_p+\chi_m = \frac{2-(1+v^2)W-u^2-2q(uvW+\frac{q}{2})}{(1-q^2)(1-u^2)},$$
$$\chi_p\chi_m = \frac{(1-W)(1-v^2W)}{(1-q^2)(1-u^2)}, \quad (4.16)$$

and ϵ becomes

$$\epsilon = \frac{\chi_m}{\chi_p}. \quad (4.17)$$

The relevant expansions of the parameters are

$$\chi_p = \chi_{p0}+(\chi_{p1}+\chi_{p2}\log(\epsilon))\epsilon, \qquad W = 1+W_1\epsilon,$$
$$v = v_0+(v_1+v_2\log(\epsilon))\epsilon, \qquad u = u_0+(u_1+u_2\log(\epsilon))\epsilon. \quad (4.18)$$

Replacing (4.17), (4.18) into (4.16), one finds the following solutions in the small ϵ limit,

[3]As we will see later on, this choice allows us to reproduce the dispersion relation in the infinite volume limit [25].

$$\chi_{p0}=\frac{1-v_0^2-u_0^2-2q(u_0v_0+\frac{q}{2})}{(1-q^2)(1-u_0^2)},$$
$$\chi_{p1}=-\frac{v_0+qu_0}{(1-q^2)^2(1-v_0^2)(1-u_0^2)^2}\left[\left(1-v_0^2-u_0^2-2q\left(u_0v_0+\frac{q}{2}\right)\right)(v_0^3+qu_0(1+3v_0^2)-v_0(1-2u_0^2-2q^2))\right.$$
$$\left.+2(1-q^2)(1-v_0^2)((1-u_0^2)v_1+(u_0v_0+q)u_1)\right]$$
$$\chi_{p2}=-\frac{2(v_0+qu_0)((1-u_0^2)v_2+(u_0v_0+q)u_2)}{(1-q^2)(1-u_0^2)^2}$$
$$W_1=-\frac{(1-v_0^2-u_0^2-2q(u_0v_0+\frac{q}{2}))^2}{(1-q^2)(1-u_0^2)(1-v_0^2)}. \tag{4.19}$$

The coefficients in the expansions of v and u will be obtained by imposing the conditions that J_2 and $\Delta\phi_1$ do not depend on ϵ, as in the cases without the B-field ($\mathrm{AdS}_5\times S^5$ and $\mathrm{AdS}_4\times CP^3$) and their TsT deformations, where the B-field is nonzero but its contribution is different.

Expanding (4.13) and (4.15) to the leading order in ϵ (now $\chi_n=K=0$), one finds that on the solutions (4.19),

$$J_2=2T\left(\frac{u_0\sqrt{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}}{1-u_0^2}+q\arcsin\sqrt{\frac{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}{(1-q^2)(1-u_0^2)}}\right), \tag{4.20}$$

$$\Delta\phi_1=2\arcsin\sqrt{\frac{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}{(1-q^2)(1-u_0^2)}}, \tag{4.21}$$

$$u_1=\frac{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}{4(1-q^2)(1-u_0^2)}[u_0(1-\log 16-v_0^2(1+\log 16))-2qv_0\log 16], \tag{4.22}$$

$$v_1=\frac{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}{4(1-q^2)(1-u_0^2)(1-v_0^2)}[v_0((1-4q^2)(1-\log 16)-u_0^2(5-\log 4096))$$
$$-v_0^3(1-\log 16-u_0^2(1+\log 16))-4qu_0(1-\log 4+v_0^2(1-\log 64))], \tag{4.23}$$

$$u_2=\frac{(u_0(1+v_0^2)+2qv_0)(1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2}))}{4(1-q^2)(1-v_0^2)}, \tag{4.24}$$

$$v_2=\frac{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}{4(1-q^2)(1-u_0^2)(1-v_0^2)}[v_0(1-v_0^2-u_0^2(3+v_0^2))-2q(u_0(1+3v_0^2)+2qv_0)]. \tag{4.25}$$

Now, let us turn to the energy-charge relation. Expanding (4.11) and (4.12) in ϵ and taking into account the solutions (4.19), (4.22)–(4.25), we obtain

$$E_s-J_1=2T\frac{\sqrt{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}}{1-u_0^2}\left(1-\frac{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}{4(1-q^2)}\epsilon\right). \tag{4.26}$$

The expression for ϵ can be found from the expansion of J_1. To the leading order, it is given by

$$\epsilon=16\exp\left[-\frac{J_1}{T}\frac{\sqrt{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}}{1-v_0^2}-2\frac{1-u_0^2-v_0^2-2q(u_0v_0+\frac{q}{2})}{(1-v_0^2)(1-u_0^2)}\right]. \tag{4.27}$$

Next, we would like to express the right-hand side of (4.26) in terms of J_2 and $\Delta\phi_1$. To this end, we solve (4.20), (4.21) with respect to u_0, v_0. The result is

$$u_0 = \frac{J_2 - qT\Delta\phi_1}{\sqrt{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}}, \tag{4.28}$$

$$v_0 = \frac{T(1-q^2)\sin\Delta\phi_1 - q(J - qT\Delta\phi_1)}{\sqrt{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}}. \tag{4.29}$$

Replacing (4.28), (4.29) into (4.26), (4.27), one finds

$$E_s - J_1 = \sqrt{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}\left(1 - \frac{(1-q^2)T^2\sin^4\frac{\Delta\phi_1}{2}}{(J_2 - qT\Delta\phi_1)^2 + 4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}\epsilon\right), \tag{4.30}$$

where

$$\epsilon = 16e^{-\frac{2\left(J_1 + \sqrt{(J_2-qT\Delta\phi_1)^2+4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}\right)\sqrt{(J_2-qT\Delta\phi_1)^2+4(1-q^2)T^2\sin^2\frac{\Delta\phi_1}{2}}\sin^2\frac{\Delta\phi_1}{2}}{(J_2-qT\Delta\phi_1)^2+4T^2\sin^4\frac{\Delta\phi_1}{2}+2qT\sin\Delta\phi_1((J_2-qT\Delta\phi_1)+\frac{q}{2}T\sin\Delta\phi_1)}}. \tag{4.31}$$

Our result[4] matches with that of [25] in (4.1) when we take the $\epsilon \to 0$ limit by sending $J_1 \to \infty$. This dispersion relation is different from the ordinary giant magnon.

The dispersion relation for the ordinary giant magnon with one nonzero angular momentum cam be obtained by setting $J_2 = 1$ and taking the limit $T \to \infty$. To take into account the *leading* finite-size effect only, we restrict ourselves to the case when $\frac{J_1}{T} >> 1$. The result is the following:

$$E_s - J_1 = T\sqrt{p^2q^2 + 4(1-q^2)\sin^2\frac{p}{2}}\left(1 - \frac{(1-q^2)\sin^4\frac{p}{2}}{p^2q^2 + 4(1-q^2)\sin^2\frac{p}{2}}\epsilon\right), \tag{4.32}$$

where

$$\epsilon = 16\exp\left[\frac{-2}{q^2(p-\sin p)^2 + 4\sin^4\frac{p}{2}}\left(\frac{J_1}{T} + \sqrt{p^2q^2 + 4(1-q^2)\sin^2\frac{p}{2}}\right)\sqrt{p^2q^2 + 4(1-q^2)\sin^2\frac{p}{2}}\sin^2\frac{p}{2}\right].$$

V. CONCLUDING REMARKS

Here we presented an approach to string dynamics in curved backgrounds with a nonzero 2-form B-field, which allows us to find the first integrals for the string coordinates along the isometric directions of the background and the corresponding conserved charges. This leads to dimensional reduction of the problem. It remains to solve the equations of motion for the nonisometric string coordinates and the Virasoro constraints. This can be done for fixed background fields. As an example we have considered string dynamics on $AdS_3 \times S^3 \times T^4$. We succeeded to find all solutions of the string equations of motion for this case, and to reduce the Virasoro constraints to algebraic relations among the embedding parameters and the integration constants. The resulting family of string configurations may have very different properties for different values of the parameters involved. That is why, we concentrated on the finite-size dyonic giant magnon solutions in this background. We have shown that the finite-size dispersion relation of (dyonic) giant magnon solution in this background is different from the analogous ones in $AdS_5 \times S^5$, $AdS_4 \times CP^3$ and their γ deformations.

Our results on the leading finite-size correction to the dispersion relation can provide an important check for the exact integrability conjecture and S-matrix elements based on it. We will report on this soon. Another possible direction of further investigation is to consider strings moving in $AdS_3 \times S^3 \times S^3 \times S^1$ which has smaller set of isometric coordinates, hence, needs to solve more nontrivial equations of motion.

[4]Equations (4.30) and (4.31) have been confirmed by an independent analysis based on the algebraic curve method [24] after our result appeared.

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APPENDIX: EXPLICIT EXACT SOLUTIONS IN $\mathrm{AdS}_3 \times S^3 \times T^4$ WITH NS-NS *B*-FIELD

Let us start with the solutions for the string coordinates in AdS_3 subspace. By using (2.17), (3.1) and (3.2), one can find that the scalar potential U_r in (3.4) is given by

$$U_r(r) = \frac{1}{2(\alpha^2-\beta^2)}\left[(\alpha\Lambda^\phi)^2 r^2 - (\alpha\Lambda^t)^2(1+r^2) + \frac{(C_\phi + q\alpha\Lambda^t r^2)^2}{r^2} - \frac{(C_t - q\alpha\Lambda^\phi r^2)^2}{1+r^2}\right]. \tag{A1}$$

After introducing the variable

$$y = r^2, \tag{A2}$$

and replacing (A1) into (3.6), one can rewrite it in the following form,

$$d\xi = \frac{\alpha^2-\beta^2}{2\alpha\sqrt{(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2]}}\frac{dy}{\sqrt{(y_p-y)(y-y_m)(y-y_n)}}, \tag{A3}$$

where

$$0 \le y_m < y < y_p, \qquad y_n < 0,$$

and y_p, y_m, y_n satisfy the relations

$$\begin{aligned}
y_p + y_m + y_n &= \frac{1}{\alpha^2(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2]}[C_r(\alpha^2-\beta^2) - \alpha(\alpha(\Lambda^\phi)^2 - 2\alpha(\Lambda^t)^2) + 2q(C_\phi\Lambda^t + C_t\Lambda^\phi) + q^2\alpha(\Lambda^t)^2],\\
y_p y_m + y_p y_n + y_m y_n &= -\frac{1}{\alpha^2(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2]}[C_r(\alpha^2-\beta^2) + C_t^2 - C_\phi^2 + \alpha^2(\Lambda^t)^2 - 2q\alpha C_\phi\Lambda^t],\\
y_p y_m y_n &= -\frac{C_\phi^2}{\alpha^2(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2]}.
\end{aligned} \tag{A4}$$

Integrating (A3) and inverting

$$\xi(y) = \frac{\alpha^2-\beta^2}{\alpha\sqrt{(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2](y_p-y_n)}}\mathbf{F}\left(\arcsin\sqrt{\frac{y_p-y}{y_p-y_m}}, \frac{y_p-y_m}{y_p-y_n}\right)$$

to $y(\xi)$, one finds the following solution,

$$y(\xi) = (y_p-y_n)\mathbf{DN}^2\left[\frac{\alpha\sqrt{(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2](y_p-y_n)}}{\alpha^2-\beta^2}\xi, \frac{y_p-y_m}{y_p-y_n}\right] + y_n, \tag{A5}$$

where $\mathbf{F}$ is the incomplete elliptic integral of the first kind and $\mathbf{DN}$ is one of the Jacobi elliptic functions.

Next, we will compute $\tilde{X}^t(\xi)$ and $\tilde{X}^\phi(\xi)$ entering (3.3). Integrating

$$\begin{aligned}
\frac{d\tilde{X}^t}{d\xi} &= \frac{1}{\alpha^2-\beta^2}\left[\beta\Lambda^t + q\alpha\Lambda^\phi - (C_t + q\alpha\Lambda^\phi)\frac{1}{1+y}\right],\\
\frac{d\tilde{X}^\phi}{d\xi} &= \frac{1}{\alpha^2-\beta^2}\left(\beta\Lambda^\phi + q\alpha\Lambda^t + \frac{C_\phi}{y}\right),
\end{aligned}$$

and using (A5), we obtain the following solutions for the string coordinates t, ϕ, in accordance with our ansatz:

$$t(\tau,\sigma) = \Lambda^t\tau + \frac{1}{\alpha\sqrt{(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2](y_p-y_n)}}\Bigg[(\beta\Lambda^t + q\alpha\Lambda^\phi)\mathbf{F}\left(\arcsin\sqrt{\frac{y_p-y}{y_p-y_m}}, \frac{y_p-y_m}{y_p-y_n}\right) - \frac{C_t+q\alpha\Lambda^\phi}{1+y_p}\Pi\left(\arcsin\sqrt{\frac{y_p-y}{y_p-y_m}}, \frac{y_p-y_m}{1+y_p}, \frac{y_p-y_m}{y_p-y_n}\right)\Bigg] \tag{A6}$$

$$\phi(\tau,\sigma) = \Lambda^\phi\tau + \frac{1}{\alpha\sqrt{(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2](y_p-y_n)}}\Bigg[(\beta\Lambda^\phi + q\alpha\Lambda^t)\mathbf{F}\left(\arcsin\sqrt{\frac{y_p-y}{y_p-y_m}}, \frac{y_p-y_m}{y_p-y_n}\right) + \frac{C_\phi}{y_p}\Pi\left(\arcsin\sqrt{\frac{y_p-y}{y_p-y_m}}, \frac{y_p-y_m}{y_p}, \frac{y_p-y_m}{y_p-y_n}\right)\Bigg], \tag{A7}$$

where Π is the incomplete elliptic integral of the third kind.

Let us compute now the string energy and spin on the solutions found. Starting from (3.8), (3.9), we obtain

$$E_s = \frac{2T}{\sqrt{(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2](y_p-y_n)}}\Bigg[\left(\Lambda^t - \frac{\beta}{\alpha^2}C_t - q\frac{C_\phi}{\alpha}\right)\mathbf{K}\left(1-\frac{y_m-y_n}{y_p-y_n}\right) + (1-q^2)\Lambda^t\left(y_n\mathbf{K}\left(1-\frac{y_m-y_n}{y_p-y_n}\right) + (y_p-y_n)\mathbf{E}\left(1-\frac{y_m-y_n}{y_p-y_n}\right)\right)\Bigg], \tag{A8}$$

$$S = \frac{2T}{\sqrt{(1-q^2)[(\Lambda^\phi)^2-(\Lambda^t)^2](y_p-y_n)}}\Bigg[\left(\frac{\beta}{\alpha^2}C_\phi + q\frac{C_t}{\alpha} + \Lambda^\phi q^2\right)\mathbf{K}\left(1-\frac{y_m-y_n}{y_p-y_n}\right) + (1-q^2)\Lambda^\phi\left(y_n\mathbf{K}\left(1-\frac{y_m-y_n}{y_p-y_n}\right) + (y_p-y_n)\mathbf{E}\left(1-\frac{y_m-y_n}{y_p-y_n}\right)\right) - \frac{q\frac{C_t}{\alpha}+q^2\Lambda^\phi}{1+y_p}\Pi\left(\frac{y_p-y_m}{1+y_p}, 1-\frac{y_m-y_n}{y_p-y_n}\right)\Bigg], \tag{A9}$$

where $\mathbf{K}$, $\mathbf{E}$ and Π are the complete elliptic integrals of the first, second and third kind.

Now we turn to the S^3 subspace. By using (2.17), (3.1) and (3.2), one can show that the scalar potential U_θ in (3.6) can be written as

$$U_\theta(\theta) = \frac{1}{2(\alpha^2-\beta^2)}\left[\frac{(C_{\phi_2} - q\alpha\Lambda^{\phi_1}\chi)^2}{\chi} + \frac{(C_{\phi_1} + q\alpha\Lambda^{\phi_2}\chi)^2}{1-\chi} + \alpha^2(\Lambda^{\phi_2})^2\chi + \alpha^2(\Lambda^{\phi_1})^2(1-\chi)\right], \tag{A10}$$

where we introduced the notation

$$\chi \equiv \cos^2\theta. \tag{A11}$$

Replacing (A10) in (3.6), one can see that it can be written in the form

$$d\xi = \frac{\alpha^2-\beta^2}{2\alpha\sqrt{(1-q^2)((\Lambda^{\phi_1})^2-(\Lambda^{\phi_2})^2)}}\frac{d\chi}{\sqrt{(\chi_p-\chi)(\chi-\chi_m)(\chi-\chi_n)}}, \tag{A12}$$

where

$$0 \le \chi_m < \chi < \chi_p \le 1, \qquad \chi_n \le 0,$$

and

$$\chi_p + \chi_m + \chi_n = \frac{1}{\alpha^2(1-q^2)((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2)}[-C_\theta(\alpha^2-\beta^2) - (\alpha\Lambda^{\phi_2})^2 + (2-q^2)(\alpha\Lambda^{\phi_1})^2 - 2q\alpha(C_{\phi_2}\Lambda^{\phi_1} + C_{\phi_1}\Lambda^{\phi_2})],$$

$$\chi_p\chi_m + \chi_p\chi_n + \chi_m\chi_n = \frac{1}{\alpha^2(1-q^2)((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2)}[(\alpha\Lambda^{\phi_1})^2 + C_{\phi_1}^2 - C_{\phi_2}^2 - C_\theta(\alpha^2-\beta^2) - 2q\alpha C_{\phi_2}\Lambda^{\phi_1}],$$

$$\chi_p\chi_m\chi_n = -\frac{(C_{\phi_2})^2}{\alpha^2(1-q^2)((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2)}.$$

Integrating (A12), one finds the following solution for χ:

$$\chi(\xi) = (\chi_p - \chi_n)\mathbf{DN}^2\left[\frac{\alpha\sqrt{(1-q^2)((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2)(\chi_p-\chi_n)}}{\alpha^2-\beta^2}\xi, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right] + \chi_n. \tag{A13}$$

Now we are ready to find the "orbit" $r = r(x)$. Written in terms of y and χ, it is given by

$$y = (y_p - y_n)\mathbf{DN}^2[\frac{\sqrt{((\Lambda^{\phi})^2 - (\Lambda^t)^2)(y_p-y_n)}}{\sqrt{((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2)(\chi_p-\chi_n)}}\mathbf{F}\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}}, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right), \frac{y_p-y_m}{y_p-y_n}] + y_n. \tag{A14}$$

Next, we compute $\tilde{X}^{\phi_1}(\xi)$ and $\tilde{X}^{\phi_2}(\xi)$. Replacing the results in our ansatz, we derive the following solutions for the isometric coordinates on S^3,

$$\phi_1 = \Lambda^{\phi_1}\tau + \frac{1}{\alpha\sqrt{(1-q^2)((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2)(\chi_p-\chi_n)}}\left[(\beta\Lambda^{\phi_1} - q\alpha\Lambda^{\phi_2})\mathbf{F}\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}}, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right) + \frac{C_{\phi_1} + q\alpha\Lambda^{\phi_2}}{1-\chi_p}\Pi\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}}, -\frac{\chi_p-\chi_m}{1-\chi_p}, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right)\right], \tag{A15}$$

$$\phi_2 = \Lambda^{\phi_2}\tau + \frac{1}{\alpha\sqrt{(1-q^2)((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2)(\chi_p-\chi_n)}}\left[(\beta\Lambda^{\phi_2} - q\alpha\Lambda^{\phi_1})\mathbf{F}\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}}, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right) + \frac{C_{\phi_1}}{\chi_p}\Pi\left(\arcsin\sqrt{\frac{\chi_p-\chi}{\chi_p-\chi_m}}, 1-\frac{\chi_m}{\chi_p}, \frac{\chi_p-\chi_m}{\chi_p-\chi_n}\right)\right]. \tag{A16}$$

Based on (3.10) and the solutions for the string coordinates on S^3 we found, we can write down the explicit expressions for the conserved angular momenta J_1 and J_2 computed on the solutions. The result is

$$J_1 = \frac{2T}{\sqrt{(1-q^2)((\Lambda^{\phi_1})^2 - (\Lambda^{\phi_2})^2)(\chi_p-\chi_n)}}\left[\left(\frac{\beta}{\alpha^2}C_{\phi_1} + \Lambda^{\phi_1} - q\frac{C_{\phi_2}}{\alpha}\right)\mathbf{K}\left(1-\frac{\chi_m-\chi_n}{\chi_p-\chi_n}\right) - (1-q^2)\Lambda^{\phi_1}\left(\chi_n\mathbf{K}\left(1-\frac{\chi_m-\chi_n}{\chi_p-\chi_n}\right) + (\chi_p-\chi_n)\mathbf{E}\left(1-\frac{\chi_m-\chi_n}{\chi_p-\chi_n}\right)\right)\right], \tag{A17}$$

$$J_2 = \frac{2T}{\sqrt{(1-q^2)((\Lambda^{\phi_1})^2-(\Lambda^{\phi_2})^2)(\chi_p-\chi_n)}}\left[\left(\frac{\beta}{\alpha^2}C_{\phi_2}-q\left(\frac{C_{\phi_1}}{\alpha}+q\Lambda^{\phi_2}\right)\right)\mathbf{K}\left(1-\frac{\chi_m-\chi_n}{\chi_p-\chi_n}\right)\right.$$
$$+(1-q^2)\Lambda^{\phi_2}\left(\chi_n\mathbf{K}\left(1-\frac{\chi_m-\chi_n}{\chi_p-\chi_n}\right)+(\chi_p-\chi_n)\mathbf{E}\left(1-\frac{\chi_m-\chi_n}{\chi_p-\chi_n}\right)\right)$$
$$\left.+\frac{q(\frac{C_{\phi_1}}{\alpha}+q\Lambda^{\phi_2})}{1-\chi_p}\Pi\left(-\frac{\chi_p-\chi_m}{1-\chi_p},1-\frac{\chi_m-\chi_n}{\chi_p-\chi_n}\right)\right]. \tag{A18}$$

Now, let us go to the T^4 subspace. Since in terms of φ^i coordinates the metric is flat and there is no B-field, the solutions for the string coordinates are simple and given by

$$\varphi^i(\tau,\sigma) = \Lambda^i\tau + \frac{1}{\alpha^2-\beta^2}(C_i+\beta\Lambda^i)\xi. \tag{A19}$$

The conserved charges (3.11) can be computed to be

$$J_i^T = \frac{2\pi\alpha T}{\alpha^2-\beta^2}\left(\frac{\beta}{\alpha}C_i+\alpha\Lambda^i\right). \tag{A20}$$

If we impose the periodicity conditions,

$$\varphi^i(\tau,\sigma)=\varphi^i(\tau,\sigma+2L)+2\pi n_i, \qquad n_i\in\mathbb{Z}_i,$$

the integration constants C_i are fixed in terms of the embedding parameters. Namely,

$$C_i=\frac{\pi n_i}{L\alpha}(\alpha^2-\beta^2)-\beta\Lambda^i. \tag{A21}$$

Replacing (A21) into (A19) and (A20), one finally finds

$$\varphi^i = \left(\Lambda^i+\frac{\beta}{\alpha}\frac{\pi n_i}{L}\right)\tau+\frac{\pi n_i}{L}\sigma,$$
$$J_i^T = 2\pi T\left(\Lambda^i+\frac{\beta}{\alpha}\frac{\pi n_i}{L}\right). \tag{A22}$$

Let us finally point out that the Virasoro constraints impose the following two conditions on the embedding parameters and integrations constants in the solutions found:

$$C_r+C_\theta=0,$$
$$\Lambda^tC_t+\Lambda^\phi C_\phi+\Lambda^{\phi_1}C_{\phi_1}+\Lambda^{\phi_2}C_{\phi_2}-\Lambda^i\left(\beta\Lambda^i-(\alpha^2-\beta^2)\frac{\pi n_i}{\alpha L}\right)=0. \tag{A23}$$

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