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Exact boundary scattering matrices of the supersymmetric sine–Gordon theory on a half-line

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Abstract. Using the boundary Yang–Baxter equations and exact results on the bulk S-matrices, we compute exact boundary scattering amplitudes of the supersymmetric sine–Gordon model with integrable boundary potentials.

1. Introduction

Quantum integrable models in two dimensions have been actively studied, since we can learn many aspects of non-perturbative physics which cannot be realized in more realistic models. These models provide an arena where one can test new ideas and get meanings of nontrivial solutions. In addition these models do provide realistic models in a limited number of applications. The quantum sine–Gordon (SG) model is the most well known exactly solvable interacting quantum field theory. Due to many theoretical developments, we now understand many nonperturbative behaviours of this model. Among them is the exact S-matrix S_{SG} of the solitons $(+)$ and antisolitons $(-)$ given by [1]

$$
\begin{aligned}
S_{++}^{++}(\theta) &= S_{--}^{--}(\theta) = U(\theta)\sinh[\lambda(\mathrm{i}\pi-\theta)] \\
S_{-+}^{+-}(\theta) &= S_{+-}^{-+}(\theta) = U(\theta)\sinh(i\pi\lambda) \\
S_{+-}^{+-}(\theta) &= S_{-+}^{-+}(\theta) = U(\theta)\sinh(\lambda\theta)
\end{aligned}
\tag{1}
$$

where $U(\theta)$ is defined by

$$
\begin{aligned}
U(\theta) &= \frac{1}{\mathrm{i}\pi}\Gamma(\lambda)\Gamma\left(1+\mathrm{i}\frac{\lambda\theta}{\pi}\right)\Gamma\left(1-\lambda-\mathrm{i}\frac{\lambda\theta}{\pi}\right)\prod_{l=1}^{\infty}\frac{F_l(\theta)F_l(\mathrm{i}\pi-\theta)}{F_l(0)F_l(\mathrm{i}\pi)} \\
F_l(\theta) &= \frac{\Gamma(2l\lambda+\mathrm{i}\frac{\lambda\theta}{\pi})\Gamma(1+2l\lambda+\mathrm{i}\frac{\lambda\theta}{\pi})}{\Gamma((2l+1)\lambda+\mathrm{i}\frac{\lambda\theta}{\pi})\Gamma(1+(2l-1)\lambda+\mathrm{i}\frac{\lambda\theta}{\pi})}
\end{aligned}
\tag{2}
$$

where $\lambda = \frac{8\pi}{\beta^2} - 1$ with β a usual coupling constant.

Recently there has been a lot of development in the study of the integrable models on the half-line or other restricted domain of the 1D space. The main motivation is that these models can be applied to 3D spherically symmetric physical systems where the s-wave element becomes dominant. The one-channel Kondo problem and the monopole-catalysed

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proton decay are frequently cited examples of these. The quantum SG model on the half-line preserves the integrability if the boundary potential is given by [2]

$$\mathcal{L}_{\rm b} = \Lambda \cos\left(\frac{\beta(\phi - \phi_0)}{2}\right). \tag{3}$$

The integrability makes it possible to describe this theory as a factorizable scattering theory of the solitons and their bound states.

Due to the existence of the boundary, we should introduce one more scattering amplitude, the boundary scattering amplitudes $R_a^b(\theta)$, in addition to the bulk scattering. For the bulk, the multi-particle scattering amplitudes are factorized into a product of two-particle S-matrices if they satisfy the Yang–Baxter equation (YBE) as a consistency condition. For the boundary scattering, where particles scatter off the boundary, we need a new type of consistency condition, namely the boundary Yang–Baxter equation (BYBE) (also known as the reflection equation), which can be expressed as

$$\begin{aligned}\sum_{c_1,c_2,d_1,d_2} & S_{a_1a_2}^{c_1c_2}(\theta_1-\theta_2) R_{c_1}^{d_1}(\theta_1) S_{c_2d_1}^{d_2b_1}(\theta_1+\theta_2) R_{d_2}^{b_2}(\theta_2) \\ &= \sum_{c_1,c_2,d_1,d_2} R_{a_2}^{c_2}(\theta_2) S_{a_1c_2}^{c_1d_2}(\theta_1+\theta_2) R_{c_1}^{d_1}(\theta_2) S_{d_2d_1}^{b_2b_1}(\theta_1-\theta_2).\end{aligned} \tag{4}$$

Besides the YBE, we need the unitarity and crossing-symmetry requirements to fix the scattering amplitudes completely up to CDD ambiguity. For the boundary scattering, this condition is expressed as the boundary cross-unitarity condition,

$$R_{\bar a}^{b}\left(\frac{\mathrm{i}\pi}{2}-\theta\right) = S_{a'b'}^{ab}(2\theta) R_{\bar b'}^{a'}\left(\frac{\mathrm{i}\pi}{2}+\theta\right). \tag{5}$$

For later use, we summarize the known results of the boundary SG model (A, $\overline{A}$ and B stand for soliton, antisoliton, and the boundary, respectively):

$$\begin{aligned} &A(\theta)B = P_+(\theta)A(-\theta)B + Q_+(\theta)\overline{A}(-\theta)B \\ &\overline{A}(\theta)B = P_-(\theta)\overline{A}(-\theta)B + Q_-(\theta)A(-\theta)B \\ &P_\pm(\theta) = \cos(\xi \mp \mathrm{i}\lambda\theta)R(u) \qquad Q_\pm = -\frac{k}{2}\sin(2\mathrm{i}\lambda\theta)R(u)\end{aligned} \tag{6}$$

where the prefactor is given by $R(\theta) = R_0(\theta)R_1(\theta)$ with

$$\begin{aligned} R_0(\theta) &= \frac{\Gamma\left(1+2\mathrm{i}\frac{\lambda\theta}{\pi}\right)\Gamma\left(\lambda-2\mathrm{i}\frac{\lambda\theta}{\pi}\right)}{\Gamma\left(1+2\mathrm{i}\frac{\lambda\theta}{\pi}\right)\Gamma\left(\lambda-2\mathrm{i}\frac{\lambda\theta}{\pi}\right)}\prod_{k=1}^{\infty}\frac{F_{2k}(2\theta)}{F_{2k}(-2\theta)} \\ R_1(\theta) &= \frac{1}{\cos\xi}\sigma(\eta,-\mathrm{i}\theta)\sigma(\mathrm{i}\vartheta,-\mathrm{i}\theta)\end{aligned} \tag{7}$$

where F_k is given in equation (2) and

$$\begin{aligned} \sigma(x,u) &= \frac{\Pi(x,\frac{\pi}{2}-u)\Pi(-x,\frac{\pi}{2}-u)\Pi(x,-\frac{\pi}{2}+u)\Pi(-x,-\frac{\pi}{2}+u)}{\Pi^2(x,\frac{\pi}{2})\Pi^2(-x,\frac{\pi}{2})} \\ \Pi(x,u) &= \prod_{l=0}^{\infty}\frac{\Gamma[\frac{1}{2}+(2l+\frac{1}{2})\lambda+\frac{x}{\pi}-\frac{\lambda u}{\pi}]\Gamma[\frac{1}{2}+(2l+\frac{3}{2})\lambda+\frac{x}{\pi}]}{\Gamma[\frac{1}{2}+(2l+\frac{3}{2})\lambda+\frac{x}{\pi}-\frac{\lambda u}{\pi}]\Gamma[\frac{1}{2}+(2l+\frac{1}{2})\lambda+\frac{x}{\pi}]}.\end{aligned} \tag{8}$$

The parameters η, ϑ are related to k, ξ by

$$\cos\eta\cos\vartheta = -\frac{1}{k}\cos\xi \qquad \cos^2\eta + \cos^2\vartheta = 1 + \frac{1}{k^2}. \tag{9}$$

The relationship between these parameters and those in equation (3), M, ϕ_0, is not completely understood [2].

2. Supersymmetric sine–Gordon theory without boundary

Our objective in this paper is to solve the $N = 1$ SUSY sine–Gordon (SSG) theory on the half-line with an appropriate boundary potential which preserves the integrability. The action of the SSG model is given by [4]

$$S = \int \mathrm{d}x\,\mathrm{d}t\,\left[\frac{1}{2}(\partial_\mu\phi)^2 - \mathrm{i}\bar{\psi}\,\partial\!\!\!/\psi - \frac{m^2}{\beta^2}\cos^2\phi - 2m\left(\cos\frac{\beta\phi}{2}\right)\bar{\psi}\psi\right] \tag{10}$$

where ϕ is a real scalar field and ψ is a Majorana fermion. β is a coupling constant of the SG theory and m is the mass parameter denoting the deviation from the massless theory. The SSG theory is integrable because it is equivalent to Toda theory on the twisted super affine Lie algebra $C^{(2)}(2)$ [5]. The equation of motion of the SSG theory can be written as a super zero-curvature condition. An infinite number of conserved charges at the classical level were derived and seem to be preserved at the lowest order of quantum level. Due to the integrability, there exist solitons and anti-solitons, as well as their bound states. All these particles form a SUSY multiplet. In this model the SUSY algebra is extended by the central charge which is the topological charge of the soliton and the antisoliton [3].

Exact results on the SSG theory have been derived due to the development of the perturbed CFTs. It has been well known that the S-matrix of the minimal CFTs, $\mathcal{M}_{p/p+1}$, with the central charge $c = 1 - \frac{6}{p(p+1)}$ perturbed by the least relevant operator can be obtained from the SG theory by restricting the S-matrix to the RSOS type using the hidden quantum group symmetry [7, 8]. The particle spectrum of this so-called restricted SG (RSG) theory is composed of the kinks K_{ab} which connect two adjacent spins a, b with $|a-b| = \frac{1}{2}$, instead of the (anti)solitons.

The S-matrix of the RSG theory, $S^{(p)}_{\mathrm{RSG}}$, is given by

$$S^{ab}_{dc}(\theta) = U(\theta)(X^{ab}_{cd})^{-\frac{\theta}{2\pi\mathrm{i}}}\left[\sqrt{X^{ab}_{cd}}\sinh\left(\frac{\theta}{p}\right)\delta_{db} + \sinh\left(\frac{\mathrm{i}\pi-\theta}{p}\right)\delta_{ac}\right] \tag{11}$$

for the process $|K_{da_1}(\theta_1)\rangle + |K_{a_2b}(\theta_2)\rangle \to |K_{dc_1}(\theta_2)\rangle + |K_{c_2b}(\theta_1)\rangle$ where

$$X^{ab}_{cd} = \left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]}\right)$$

with q-number $[n] = (q^n - q^{-n})/(q - q^{-1})$ and $q = -\mathrm{e}^{-\mathrm{i}\pi/p}$. The allowed values of spins are $0, \frac{1}{2}, 1, \ldots, \frac{p}{2} - 1$.

The exact S-matrix of the SSG theory has been obtained as a by-product from the result of the perturbed super CFT; the perturbed super CFTs have the S-matrix in the factorized form

$$S_{\mathrm{SCFT}}(\theta) = S^{(4)}_{\mathrm{RSG}}(\theta) \otimes S^{(p)}_{\mathrm{RSG}}(\theta) \tag{12}$$

and by 'unrestricting' $S^{(p)}_{\mathrm{RSG}}$, one obtains the SSG (anti)soliton S-matrix [9]:

$$S_{\mathrm{SSG}}(\theta) = S^{(4)}_{\mathrm{RSG}}(\theta) \otimes S_{\mathrm{SG}}(\theta). \tag{13}$$

The first S-matrix factor which commutes with the SUSY charges, applies to the superspace indices of the SSG particles and the second one is nothing but the SG (anti)soliton S-matrix, equation (1), however, with different parameterization

$$\lambda = \frac{2\pi}{\beta^2} - \frac{1}{2}. \tag{14}$$

By denoting the SG solitons with topological charge ± 1 by $|A^\pm\rangle$, the particle states of the SSG theory can be denoted by $|K^\pm_{ab}\rangle = |K_{ab}\rangle \otimes |A^\pm\rangle$, where the first quantum number carries the SUSY charges and the second the topological charges. Explicit SUSY transformations are as follows [10]:

$$\begin{aligned} Q|K^{\pm}_{0\frac{1}{2}}\rangle &= -\mathrm{i}\mathrm{e}^{\theta/2}|K^{\pm}_{1\frac{1}{2}}\rangle \qquad & \overline{Q}|K^{\pm}_{0\frac{1}{2}}\rangle &= \mp\mathrm{i}\mathrm{e}^{-\theta/2}|K^{\pm}_{1\frac{1}{2}}t\rangle \\ Q|K^{\pm}_{1\frac{1}{2}}\rangle &= \mathrm{i}\mathrm{e}^{\theta/2}|K^{\pm}_{0\frac{1}{2}}\rangle \qquad & \overline{Q}|K^{\pm}_{1\frac{1}{2}}\rangle &= \pm\mathrm{i}\mathrm{e}^{-\theta/2}|K^{\pm}_{0\frac{1}{2}}\rangle \end{aligned} \tag{15}$$

and on the charge conjugated states by

$$\begin{aligned} Q|K^{\pm}_{\frac{1}{2}0}\rangle &= \mathrm{e}^{\theta/2}|K^{\pm}_{\frac{1}{2}0}\rangle \qquad & \overline{Q}|K^{\pm}_{\frac{1}{2}0}\rangle &= \pm\mathrm{e}^{-\theta/2}|K^{\pm}_{\frac{1}{2}0}\rangle \\ Q|K^{\pm}_{\frac{1}{2}1}\rangle &= -\mathrm{e}^{\theta/2}|K^{\pm}_{\frac{1}{2}1}\rangle \qquad & \overline{Q}|K^{\pm}_{\frac{1}{2}1}\rangle &= \mp\mathrm{e}^{-\theta/2}|K^{\pm}_{\frac{1}{2}1}\rangle. \end{aligned} \tag{16}$$

From these relations, one can check that the SUSY charges satisfy

$$Q^2 = P = \mathrm{e}^{\theta} \qquad \overline{Q}^2 = \overline{P} = \mathrm{e}^{-\theta} \qquad \text{and} \qquad Q\overline{Q} + \overline{Q}Q = 2T. \tag{17}$$

The central charge T is ± 1 corresponding to the topological charges of the SSG solitons.

3. Boundary scattering matrices of the supersymmetry sine–Gordon theory

Integrability is often preserved by the introduction of the SUSY. Therefore, one can naturally guess that the SUSY extension of the SG theory on the half-line can be integrable. In recent work, it has been claimed that the half-line SSG theory is integrable if one introduces a well-defined boundary potential. In [6], it has been shown that the SSG model can preserve integrability with the following boundary potential:

$$\mathcal{L}_{\mathrm{b}} = \Lambda \cos\left(\frac{\beta(\phi - \phi_0)}{2}\right) + M\bar{\psi}\psi + \epsilon\psi + \bar{\epsilon}\bar{\psi}. \tag{18}$$

This potential, however, does not preserve supersymmetry on the bulk. In fact, one can at most restore half of the bulk supersymmetry, $Q \pm \overline{Q}$. This is obtained by tuning the parameters of the boundary potential to following values:

$$\Lambda = \pm\frac{2m}{\beta} \qquad \phi_0 = 0 \qquad M = \pm 1 \qquad \epsilon = \bar{\epsilon} = 0.$$

In this paper, we shall be mainly concerned with integrability of the SSG model on the half-line and not yet be concerned with the preservation of the bulk supersymmetry. Due to this integrability, we can describe the boundary SSG model as a scattering theory where the amplitudes can be obtained from the BYBE.

Since the bulk SSG S-matrix has a factorized form, we will restrict ourselves to finding the boundary scattering R-matrix in the factorized form

$$R_{\mathrm{SSG}}(\theta) = R_{\mathrm{SUSY}}(\theta) \otimes R_{\mathrm{SG}}(\theta) \tag{19}$$

as well. So each factor satisfies the boundary YBE, equation (4), separately. The second factor is the usual SG part, equation (6), with λ given by equation (14). The first factor is what we are going determine based on the boundary YBE.

This boundary scattering matrix satisfies the boundary YBE in the RSOS representation given by [11–13]

$$\begin{aligned} &\sum_{a_1,b_1} R^{a}_{bb_1}(\theta_1) S^{ac}_{b_1a_1}(\theta_2+\theta_1) R^{a_1}_{b_1b_2}(\theta_2) S^{a_1c}_{b_2a_2}(\theta_2-\theta_1) \\ &\qquad = \sum_{a_1,b_1} S^{ac}_{ba_1}(\theta_2-\theta_1) R^{a_1}_{bb_1}(\theta_2) S^{a_1c}_{b_1a_2}(\theta_2+\theta_1) R^{a_2}_{b_1b_2}(\theta_1) \end{aligned} \tag{20}$$

where $R^a_{bc}(\theta)$ denotes the boundary S-matrix and the bulk scattering matrix $S^{ab}_{cd}(\theta)$ is given by equation (11).

In general $R^a_{bc}(\theta)$ contains both diagonal and off-diagonal scattering components, which can be written as

$$R^a_{bc}(\theta) = R(\theta)(X^{bc}_{aa})^{-\frac{\theta}{2\pi i}}[\delta_{b\neq c}X^a_{bc}(\theta) + \delta_{bc}(\delta_{b-1/2,a}U_a(\theta) + \delta_{b+1/2,a}D_a(\theta))] \tag{21}$$

where $R(\theta)$ have to be determined from the boundary crossing and unitarity constraints, while X^a_{bc} and U_a, D_a have to be determined from the BYBE. An overall q-number factor is multiplied by the above to cancel that from the bulk S-matrix in order to simplify the BYBE. For $p = 4$, there are three RSOS spins labelled by 0, $\frac{1}{2}$, and 1. The unknown scattering amplitudes are $U_0, D_1, U_{\frac{1}{2}}, D_{\frac{1}{2}}$ for diagonal scattering weights and $X^{\frac{1}{2}}_{01}, X^{\frac{1}{2}}_{10}$ for off-diagonal scattering weights.

Substituting equation (21) into the BYBE, one finds that the unknowns satisfy the following equations:

$$\begin{aligned}
&X^{\frac{1}{2}}_{01}(\theta')X^{\frac{1}{2}}_{10}(\theta) = X^{\frac{1}{2}}_{01}(\theta)X^{\frac{1}{2}}_{10}(\theta')\\
&U_{\frac{1}{2}}(\theta)(1+\sqrt{2}f_-) + D_{\frac{1}{2}}(\theta')(1+\sqrt{2}f_+)(1+\sqrt{2}f_-) = U_{\frac{1}{2}}(\theta') + D_{\frac{1}{2}}(\theta)(1+\sqrt{2}f_+)\\
&D_{\frac{1}{2}}(\theta)(1+\sqrt{2}f_-) + D_{\frac{1}{2}}(\theta')(1+\sqrt{2}f_+) = (1+\sqrt{2}f_-)D_{\frac{1}{2}}(\theta') + U_{\frac{1}{2}}(\theta)(1+\sqrt{2}f_+)\\
&U_0(\theta')D_1(\theta)f_+\left(1+\frac{f_-}{\sqrt{2}}\right) + D_1(\theta')D_1(\theta)f_-\left(1+\frac{f_+}{\sqrt{2}}\right)\\
&\qquad = U_0(\theta)D_1(\theta')f_+\left(1+\frac{f_-}{\sqrt{2}}\right) + U_0(\theta')U_0(\theta)f_-\left(1+\frac{f_+}{\sqrt{2}}\right)
\end{aligned} \tag{22}$$

where

$$f_- = \frac{\sinh(\frac{\theta'-\theta}{4})}{\sinh(\frac{\mathrm{i}\pi-\theta'+\theta}{4})} \qquad f_+ = \frac{\sinh(\frac{\theta'+\theta}{4})}{\sinh(\frac{\mathrm{i}\pi-\theta'-\theta}{4})}.$$

It is important to point out that $U_{\frac{1}{2}}$ (U_0) is coupled to $D_{\frac{1}{2}}$ (D_1) through the BYBE, and the two equations relating $U_{\frac{1}{2}}, D_{\frac{1}{2}}$ are there only if $X^{\frac{1}{2}}_{01}, X^{\frac{1}{2}}_{10}$ are nonvanishing. If the later are taken to be zero in the first place, i.e. off-diagonal scattering is forbidden, then the BYBE does not provide any information on $U_{\frac{1}{2}}, D_{\frac{1}{2}}$, a case we will elaborate later.

From equation (22), we have

$$X^{\frac{1}{2}}_{01}(\theta) \propto X^{\frac{1}{2}}_{10}(\theta). \tag{23}$$

The constant of proportionality can actually be shown to be a gauge factor, hence the difference between these two off-diagonal scattering amplitudes is not significant at this point. While the rest of the equations can be turned into ordinary differential equations in the limit $\theta' \to \theta$, giving respectively

$$\begin{aligned}
&[\dot{U}_{\frac{1}{2}}(\theta) + \dot{D}_{\frac{1}{2}}(\theta)]\tanh\frac{\theta}{2} + 2[U_{\frac{1}{2}}(\theta) + D_{\frac{1}{2}}(\theta)] = 0\\
&[\dot{U}_{\frac{1}{2}}(\theta) - \dot{D}_{\frac{1}{2}}(\theta)]\coth\frac{\theta}{2} - 2[U_{\frac{1}{2}}(\theta) - D_{\frac{1}{2}}(\theta)] = 0
\end{aligned} \tag{24}$$

and

$$\dot{\mathcal{R}}(\theta)\tanh\frac{\theta}{2} = \mathcal{R}(\theta)^2 - 1 \tag{25}$$

where $\mathcal{R}(\theta) \equiv D_1(\theta)/U_0(\theta)$ and $\dot{}$ denotes differentiation with respect to θ. These equations can be easily integrated to give

$$\begin{aligned} U_{\frac{1}{2}}(\theta) &= \frac{B}{\sinh\frac{\theta}{2}} + C\cosh\frac{\theta}{2} \\ D_{\frac{1}{2}}(\theta) &= \frac{B}{\sinh\frac{\theta}{2}} - C\cosh\frac{\theta}{2} \\ \frac{D_1(\theta)}{U_0(\theta)} &= \frac{1 - A\sinh\frac{\theta}{2}}{1 + A\sinh\frac{\theta}{2}} \end{aligned} \tag{26}$$

where A, B, C are free parameters.

Next, we want to determine the overall $R(\theta)$ with the use of the boundary unitarity and crossing symmetry conditions which can be summarized as

$$\sum_c R_{bc}^a(\theta) R_{cd}^a(-\theta) = \delta_{bd} \qquad \sum_d S_{bd}^{ac}(2\theta) R_{bc}^d\left(\frac{\mathrm{i}\pi}{2} + \theta\right) = R_{bc}^a\left(\frac{\mathrm{i}\pi}{2} - \theta\right). \tag{27}$$

First we consider the common $R(\theta)$ factor of the weights $X_{10}^{\frac{1}{2}}, X_{01}^{\frac{1}{2}}, U_2, D_2$. The above two conditions give respectively

$$R(\theta)R(-\theta)\left(X_{01}^{\frac{1}{2}}X_{10}^{\frac{1}{2}} + C^2\cosh^2\frac{\theta}{2} - \frac{B^2}{\sinh^2\frac{\theta}{2}}\right) = 1 \tag{28}$$

and

$$U(2\theta)R\left(\frac{\mathrm{i}\pi}{2} + \theta\right)\sinh\left(\frac{\mathrm{i}\pi}{4} - \frac{\theta}{2}\right) = R\left(\frac{\mathrm{i}\pi}{2} - \theta\right). \tag{29}$$

To arrive at the above, use has been made of the following relations:

$$\begin{aligned} D_{\frac{1}{2}}(\theta) &= D_{\frac{1}{2}}(\mathrm{i}\pi - \theta)\left[1 + \frac{\sqrt{2}\sinh(\frac{\mathrm{i}\pi-2\theta}{4})}{\sinh\frac{\theta}{2}}\right] \\ U_{\frac{1}{2}}(\theta) &= U_{\frac{1}{2}}(\mathrm{i}\pi - \theta)\left[1 + \frac{\sqrt{2}\sinh(\frac{\mathrm{i}\pi-2\theta}{4})}{\sinh\frac{\theta}{2}}\right] \end{aligned} \tag{30}$$

which can be obtained from equation (22), taking the limit $\theta' \to \mathrm{i}\pi - \theta$. In the unitarity condition, the non-zero factors $X_{01}^{\frac{1}{2}}$ and $X_{10}^{\frac{1}{2}}$ can be absorbed into $R(\theta)$ and we set it as -1 for convenience.

To solve for $R(\theta)$, we write

$$R(\theta) = \sinh\frac{\theta}{2} R_0(\theta) R_1(\theta)$$

where now

$$R_0(\theta)R_0(-\theta) = 1 \qquad U(2\theta)R_0\left(\frac{\mathrm{i}\pi}{2} + \theta\right)\sinh\left(\frac{\mathrm{i}\pi}{4} + \frac{\theta}{2}\right) = R_0\left(\frac{\mathrm{i}\pi}{2} - \theta\right) \tag{31}$$

whose minimal solution does not depend on the free parameters B, C.

While R_1 contains all the information of the boundary conditions and satisfies

$$R_1(\theta)R_1(-\theta)\left[B^2 + (1 - C^2)\sinh^2\frac{\theta}{2} - C^2\sinh^4\frac{\theta}{2}\right] = 1 \qquad R_1(\theta) = R_1(\mathrm{i}\pi - \theta) \tag{32}$$

the minimal solution is given by equation (7) with the following replacements:

$$\cos^2\xi \to B^2 \qquad k^2 \to -C^2 \qquad \lambda \to \tfrac{1}{2}.$$

The $R(\theta)$ factor of the weights U_0, D_1 need not be the same as that determined above since as mentioned before these weights are not coupled to those treated before. The unitarity condition gives

$$R(\theta)R(-\theta)U_0(\theta)U_0(-\theta)=1 \qquad R(\theta)R(-\theta)D_1(\theta)D_1(-\theta)=1$$

while the crossing symmetry gives

$$\begin{aligned} U(\mathrm{i}\pi-2\theta)R(\mathrm{i}\pi-\theta)&\left[U_0(\mathrm{i}\pi-\theta)\sinh\left(\frac{2\theta+\mathrm{i}\pi}{4}\right)+D_1(\mathrm{i}\pi-\theta)\sinh\left(\frac{\mathrm{i}\pi-2\theta}{4}\right)\right]\\ &=\sqrt{2}R(\theta)U_0(\theta)\\ U(\mathrm{i}\pi-2\theta)R(\mathrm{i}\pi-\theta)&\left[D_1(\mathrm{i}\pi-\theta)\sinh\left(\frac{\mathrm{i}\pi-2\theta}{4}\right)+U_0(\mathrm{i}\pi-\theta)\sinh\left(\frac{\mathrm{i}\pi-2\theta}{4}\right)\right]\\ &=\sqrt{2}R(\theta)D_1(\theta). \end{aligned} \tag{33}$$

We can solve these equations separately by requiring

$$R(\theta)R(-\theta)=1 \qquad U(2\theta)R\left(\frac{\mathrm{i}\pi}{2}+\theta\right)\sinh\left(\frac{\mathrm{i}\pi}{4}-\frac{\theta}{2}\right)=R\left(\frac{\mathrm{i}\pi}{2}-\theta\right) \tag{34}$$

so that $R(\theta)$ is given by $R_0(\theta)$ in equation (7) with $\lambda=\frac{1}{4}$. While U_0, D_1 satisfy

$$U_0(\theta)U_0(-\theta)=1 \qquad D_1(\theta)D_1(-\theta)=1 \tag{35}$$

and

$$\begin{aligned} U_0(\mathrm{i}\pi-\theta)\sinh\left(\frac{2\theta+\mathrm{i}\pi}{4}\right)+D_1(\mathrm{i}\pi-\theta)\sinh\left(\frac{\mathrm{i}\pi-2\theta}{4}\right)&=\sqrt{2}U_0(\theta)\sinh\frac{\theta}{2}\\ D_1(\mathrm{i}\pi-\theta)\sinh\left(\frac{\mathrm{i}\pi-2\theta}{4}\right)+U_0(\mathrm{i}\pi-\theta)\sinh\left(\frac{\mathrm{i}\pi-2\theta}{4}\right)&=\sqrt{2}D_1(\theta)\sinh\frac{\theta}{2}. \end{aligned}$$

These two sets of equations are compatible with equations (26), (22), hence they contain only two independent relations. Substituting the ratio of D_1, U_0 into the above we get a relation between $U_0(\theta)$ ($D_1(\theta)$) and $U_0(\mathrm{i}\pi-\theta)$ ($D_1(\mathrm{i}\pi-\theta)$):

$$\begin{aligned} \frac{U_0(\theta)}{U_0(\mathrm{i}\pi-\theta)}&=\frac{\mathrm{i}\cosh\frac{\theta}{2}(1+A\sinh\frac{\theta}{2})}{\sinh\frac{\theta}{2}(1+\mathrm{i}A\cosh\frac{\theta}{2})}\\ \frac{D_1(\theta)}{D_1(\mathrm{i}\pi-\theta)}&=\frac{\mathrm{i}\cosh\frac{\theta}{2}(1-A\sinh\frac{\theta}{2})}{\sinh\frac{\theta}{2}(1-\mathrm{i}A\cosh\frac{\theta}{2})}. \end{aligned}$$

These relations together with equation (35) can determine $U_0(\theta)$ and $D_1(\theta)$ up to the CDD factor:

$$U_0(\theta)=\left(\frac{\sinh\frac{\mathrm{i}\Delta}{2}}{\sinh\frac{\theta}{2}}+1\right)R(\theta)R(\mathrm{i}\pi-\theta) \qquad D_1(\theta)=\left(\frac{\sinh\frac{\mathrm{i}\Delta}{2}}{\sinh\frac{\theta}{2}}-1\right)R(\theta)R(\mathrm{i}\pi-\theta)$$

$$R(\theta)=\frac{\Gamma(\frac{-\mathrm{i}\theta}{2\pi})}{\Gamma(\frac{1}{2}-\frac{\mathrm{i}\theta}{2\pi})}\prod_{l=1}^{\infty}\frac{\Gamma(\frac{\Delta}{2\pi}-\frac{\mathrm{i}\theta}{2\pi}+l)\Gamma(-\frac{\mathrm{i}\theta}{2\pi}-\frac{\Delta}{2\pi}+l-1)\Gamma^2(-\frac{\mathrm{i}\theta}{2\pi}+l-\frac{1}{2})}{\Gamma(\frac{\Delta}{2\pi}-\frac{\mathrm{i}\theta}{2\pi}+l+\frac{1}{2})\Gamma(-\frac{\mathrm{i}\theta}{2\pi}-\frac{\Delta}{2\pi}+l-\frac{1}{2})\Gamma^2(-\frac{\mathrm{i}\theta}{2\pi}+l-1)}$$

where $A^{-1}=\mathrm{i}\sin\frac{\Delta}{2}$.

Finally, we consider the case where $X_{10}^{\frac{1}{2}}$, $X_{01}^{\frac{1}{2}}$ are zero to begin with. Then unitarity and crossing symmetry are the only conditions that can be used to determined $U_{\frac{1}{2}}$, $D_{\frac{1}{2}}$. These conditions give

$$R(\theta)R(-\theta)U_{\frac{1}{2}}(\theta)U_{\frac{1}{2}}(-\theta)=1 \qquad R(\theta)R(-\theta)D_{\frac{1}{2}}(\theta)D_{\frac{1}{2}}(-\theta)=1.$$

and

$$U(\mathrm{i}\pi-2\theta)R(\mathrm{i}\pi-\theta)U_{\frac{1}{2}}(\mathrm{i}\pi-\theta)\sinh\frac{\mathrm{i}\pi-\theta}{2}=R(\theta)U_{\frac{1}{2}}(\theta)$$
$$U(\mathrm{i}\pi-2\theta)R(\mathrm{i}\pi-\theta)D_{\frac{1}{2}}(\mathrm{i}\pi-\theta)\sinh\frac{\mathrm{i}\pi-\theta}{2}=R(\theta)D_{\frac{1}{2}}(\theta).$$

Again, we separately require $R(\theta)$ to satisfy the same relations given in equation (34), and $U_{\frac{1}{2}}, D_{\frac{1}{2}}$ to satisfy

$$U_{\frac{1}{2}}(\theta)U_{\frac{1}{2}}(-\theta)=1 \qquad D_{\frac{1}{2}}(\theta)D_{\frac{1}{2}}(-\theta)=1 \tag{36}$$

and

$$U_{\frac{1}{2}}(\mathrm{i}\pi-\theta)\sinh\frac{\mathrm{i}\pi-\theta}{2}=U_{\frac{1}{2}}(\theta)\sinh\frac{\theta}{2}$$
$$D_{\frac{1}{2}}(\mathrm{i}\pi-\theta)\sinh\frac{\mathrm{i}\pi-\theta}{2}=D_{\frac{1}{2}}(\theta)\sinh\frac{\theta}{2}.$$

From these relations, we find $U_{\frac{1}{2}}(\theta)=D_{\frac{1}{2}}(\theta)=\mathrm{i}/[\sinh(\theta/2)\sigma(\pi/2,-\mathrm{i}\theta)]$ where σ is defined in (8).

To summarize, we found two mutually exclusive sets of solutions for U_0, D_1, $U_{\frac{1}{2}}$, $D_{\frac{1}{2}}$, $X_{10}^{\frac{1}{2}}$, $X_{01}^{\frac{1}{2}}$. The first set has a non-vanishing off-diagonal element ($X_{10}^{\frac{1}{2}}\neq 0$) and introduces the undetermined parameters A, B, C, while the second set only allows diagonal scattering ($X_{10}^{\frac{1}{2}}=0$) with one free parameter A. As for the number of free parameters, we should keep in mind the further two parameters appearing in the SG soliton sector. Therefore there are in total five or three parameters for the boundary scattering theory. It should be remarked that our results are based on the assumption that the boundary scattering matrix is in the factorized form given in equation (19), so we only have proof that for the scattering amplitudes that involve U_0, D_1 are indeed factorized. It is not known whether there are other non-factorized scattering matrices that involve $U_{\frac{1}{2}}$, $D_{\frac{1}{2}}$, $X_{10}^{\frac{1}{2}}$ and $X_{01}^{\frac{1}{2}}$.

4. Discussion

Our results on the scattering matrix suggest that there are at least two integrable boundary Lagrangians for the SSG model. From the number of parameters in the theory, we claim that the non-diagonal scattering theory corresponds to the SSG model with the boundary potential given in equation (18) where five parameters are introduced. On making this claim, we have assumed that there is no other integrable boundary Lagrangian with the five parameters. On the diagonal scattering theory with three parameters, we do not know which is the boundary potential it corresponds to.

It is not clear in general how the five parameters in the scattering theory and the Lagrangian are related at this moment except for a few special cases. For $\Lambda=\infty$ with arbitrary $M,\epsilon,\bar{\epsilon}$, the topological charge is preserved, therefore topological charge violating amplitudes should be zero. Since our R-matrix has a factorized form in such a way that the SUSY sector is separated from the topological sector, $Q_\pm=0$ ($k=0$) irrespective of the SUSY sector. For $\Lambda=0$ with arbitrary $M,\epsilon,\bar{\epsilon}$, the charge conjugation symmetry is preserved, and solitons and antisolitons irrespective of their SUSY charges behave in the same way. This means $P_+=P_-$ and $Q_+=Q_-$, or $\xi=0$.

For other special cases, it is more convenient to re-express the SUSY sector in terms of the SUSY eigenstate, $|B^\pm\rangle$ and $|F^\pm\rangle$ defined by

$$|B^\pm\rangle=\frac{1}{\sqrt{2}}(|K^\pm_{0\frac{1}{2}}\rangle+|K^\pm_{1\frac{1}{2}}\rangle) \qquad |F^\pm\rangle=\frac{1}{\sqrt{2}}(|K^\pm_{0\frac{1}{2}}\rangle-|K^\pm_{1\frac{1}{2}}\rangle)$$

$$|\overline{B}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|K^{\pm}_{\frac{1}{2}0}\rangle + |K^{\pm}_{\frac{1}{2}1}\rangle) \qquad |\overline{F}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|K^{\pm}_{\frac{1}{2}0}\rangle - |K^{\pm}_{\frac{1}{2}1}\rangle).$$

Now solitons and antisolitons carry a well-defined fermion number F, say, $F = 0$ for $|B^{\pm}\rangle$ and $F = 1$ for $|F^{\pm}\rangle$. If we rewrite the boundary R-matrix in this basis, one finds

$$\begin{aligned}
B^{\pm}(\theta)B &= \tfrac{1}{2}(D_{\frac{1}{2}} + U_{\frac{1}{2}} + X_{+})B^{\pm}(-\theta)B + \tfrac{1}{2}(D_{\frac{1}{2}} - U_{\frac{1}{2}} + X_{-})F^{\pm}(-\theta)B \\
F^{\pm}(\theta)B &= \tfrac{1}{2}(D_{\frac{1}{2}} - U_{\frac{1}{2}} - X_{-})B^{\pm}(-\theta)B + \tfrac{1}{2}(D_{\frac{1}{2}} + U_{\frac{1}{2}} - X_{+})F^{\pm}(-\theta)B \\
\overline{B}^{\pm}(\theta)B &= \tfrac{1}{2}(U_0 + D_1)\overline{B}^{\pm}(-\theta)B + \tfrac{1}{2}(U_0 - D_1)\overline{F}^{\pm}(-\theta)B \\
\overline{F}^{\pm}(\theta)B &= \tfrac{1}{2}(U_0 - D_1)\overline{B}^{\pm}(-\theta)B + \tfrac{1}{2}(U_0 + D_1)\overline{F}^{\pm}(-\theta)B
\end{aligned} \tag{37}$$

where $X_{\pm} = X^{\frac{1}{2}}_{10} \pm X^{\frac{1}{2}}_{01}$.

Now consider the case where $\epsilon = \bar{\epsilon} = 0$. Since the Lagrangian preserves the fermion number, the fermion number violating amplitudes in the R-matrix should vanish. By choosing a gauge, we can fix $X^{\frac{1}{2}}_{01} = X^{\frac{1}{2}}_{10}$. This leaves $D_{\frac{1}{2}} = U_{\frac{1}{2}}$ and $U_0 = D_1$, which means $A = C = 0$ if $\epsilon = \bar{\epsilon} = 0$. For more complete relations, it is desirable to consider the boundary scattering of the SSG bound states since the lowest massive bound states are ϕ and ψ fields appearing in the Lagrangian. One can use the bootstrap procedure for this computation, although we will not pursue it here. With the S-matrices of the bound states, one can examine the suspersymmetry aspects of the model in terms of the free parameters and make comparison with that found in [6]. We hope to report on this in the future.

It is interesting to note that the boundary SSG model introduces extra boundary poles in addition to those from the SG model, which in the physical strip may be interpreted as resonance states in the supersymmetric theory with a boundary.

As an extension of this work, one can also consider the *restricted* SSG model on a half-line, in which case the bulk scattering matrix has the factorized form given in equation (12), by assuming that the boundary S-matrices are also factorized, their expressions can be immediately read off from [12] as a tensor product of the boundary S-matrices of RSOS(4) and RSOS(p).

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References

[1] Zamolodchikov A and Zamolodchikov Al 1979 *Ann. Phys.* **120** 253
[2] Ghoshal S and Zamolodchikov A 1994 *Int. J. Mod. Phys.* A **9** 3841
[3] Witten E and Olive D 1978 *Phys. Lett.* **78B** 97
[4] Di Vecchia P and Ferrara S 1977 *Nucl. Phys.* B **130** 93
[5] Chaichian M and Kulish P 1987 *Phys. Lett.* **183B** 169
Babelon O and Langouche F 1987 *Nucl. Phys.* B **290** 603
Olshanesky M A 1983 *Commun. Math. Phys.* **88** 63
[6] Inami T, Odake S and Zhang Y-Z 1995 *Phys. Lett.* **359B** 118
[7] Bernard D and LeClair A 1990 *Nucl. Phys.* B **340** 721
[8] Reshetikhin N Yu and Smirnov F 1990 *Commun. Math. Phys.* **131** 157
[9] Ahn C, Bernard D and LeClair A 1990 *Nucl. Phys.* B **346** 409
[10] Ahn C 1991 *Nucl. Phys.* B **354** 57
[11] Kulish P P Yang–Baxter equations in integrable models *Preprint* hep-th/9507070

Behrend R E, Pearce P A and O'Brien D L Interaction-round-a-face models with fixed boundary conditions: the ABF fusion hierarchy *Preprint* hep-th/9507118
[12] Ahn C and Koo W M Boundary Yang–Baxter equation in the RSOS/SOS representation *Preprint* hep-th/9508080
[13] Chim L Boundary *S*-matrix for the tricritical Ising model *Preprint* hep-th/951008
[14] Shankar R and Witten E 1978 *Phys. Rev.* D **17** 2134
[15] Mussardo G, LeClair A, Saleur H and Skorik S Boundary energy and boundary states in integrable quantum theories *Preprint* hep-th/9503227