Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 798 [FS] (2008) 402–422
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

The QCD spin chain S matrix

Changrim Ahn [a,*], Rafael I. Nepomechie [b], Junji Suzuki [c]

[a] *Department of Physics, Ewha Womans University, Seoul 120-750, South Korea*
[b] *Physics Department, PO Box 248046, University of Miami, Coral Gables, FL 33124, USA*
[c] *Department of Physics, Faculty of Science, Shizuoka University, Ohya 836, Shizuoka, Japan*

Received 20 November 2007; accepted 28 December 2007

Available online 4 January 2008

Abstract

Beisert et al. have identified an integrable $SU(2,2)$ quantum spin chain which gives the one-loop anomalous dimensions of certain operators in large N_c QCD. We derive a set of nonlinear integral equations (NLIEs) for this model, and compute the scattering matrix of the various (in particular, magnon) excitations.

PACS: 05.50.+q; 11.25.Tq; 11.55.Ds

Keywords: Integrability; Nonlinear integral equation; QCD; Super-Yang–Mills theory; Anomalous dimension

1. Introduction

The search for integrability in QCD has a long history (see, e.g., [1–6] and references therein). A remarkable recent development is the discovery [7] that the one-loop mixing matrix[1] for the chiral gauge-invariant operators

$$\operatorname{tr} f_{\alpha_1\beta_1}(x)\dots f_{\alpha_L\beta_L}(x) \tag{1.1}$$

* Corresponding author.
E-mail address: ahn@ewha.ac.kr (C. Ahn).

[1] Given a set of operators $\mathcal{O}^M(x)$, the mixing matrix is defined by $\Gamma = Z^{-1} \cdot dZ/d\ln\Lambda$, where Z is the renormalization factor which makes correlation functions of $\mathcal{O}^M_{\text{ren}}(x) = Z^M_N \mathcal{O}^N(x)$ finite, and Λ is the ultraviolet cutoff. See also [8].

doi:10.1016/j.nuclphysb.2007.12.026

in the limit $N_c \to \infty$ is given by the integrable spin-1 antiferromagnetic XXX Hamiltonian [9,10],

$$\Gamma = \frac{\alpha_s N_c}{2\pi} \sum_{l=1}^{L} \left[\frac{7}{6} + \frac{1}{2} \vec{S}_l \cdot \vec{S}_{l+1} - \frac{1}{2} (\vec{S}_l \cdot \vec{S}_{l+1})^2 \right]. \tag{1.2}$$

Here $f_{\alpha\beta}$ are the selfdual components of the Yang–Mills field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_{\text{YM}} [A_\mu, A_\nu]$ (where the gauge fields $A_\mu(x)$ are $N_c \times N_c$ Hermitian matrices), which together with the anti-selfdual components $\bar{f}_{\dot\alpha\dot\beta}$ are defined by

$$F_{\mu\nu} = \sigma^{\alpha\beta}_{\mu\nu} f_{\alpha\beta} + \bar{\sigma}^{\dot\alpha\dot\beta}_{\mu\nu} \bar{f}_{\dot\alpha\dot\beta}, \tag{1.3}$$

where $\sigma_{\mu\nu} = i\sigma_2(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)/4$, $\bar{\sigma}_{\mu\nu} = -i(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu)\sigma_2/4$ and $\sigma_\mu = (1, \vec{\sigma})$, $\bar{\sigma}_\mu = (1, -\vec{\sigma})$. Moreover, $\alpha_s = g^2_{YM}/4\pi$, $\alpha_s N_c$ is the 't Hooft coupling [1] which is assumed to be small, and $\vec{S}$ are spin-1 generators of $SU(2)$. Indeed, since $f_{\alpha\beta}$ has three independent components

$$f_+ = f_{11}, \qquad f_0 = \frac{1}{\sqrt{2}}(f_{12} + f_{21}), \qquad f_- = f_{22}, \tag{1.4}$$

the operators (1.1) can be identified with the Hilbert space of a periodic spin-1 quantum spin chain of length L. The eigenvectors and eigenvalues of Γ, i.e., the linear combinations of the operators (1.1) which are multiplicatively renormalizable and their anomalous dimensions, respectively, can therefore be obtained using the Bethe ansatz [11,12]. In particular, the anomalous dimensions are given by

$$\gamma = \frac{\alpha_s N_c}{2\pi} \left(\frac{7L}{6} - \sum_{j=1}^{M_l} \frac{2}{l_j^2 + 1} \right), \tag{1.5}$$

where $\{l_1, \ldots, l_{M_l}\}$ are roots of the Bethe ansatz equations (BAEs)[2]

$$\left(\frac{l_j + i}{l_j - i} \right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{M_l} \frac{l_j - l_k + i}{l_j - l_k - i}. \tag{1.6}$$

This result was generalized in [13] to gauge-invariant operators with derivatives

$$\operatorname{tr}\big(D^{m_1} f\big) \ldots \big(D^{m_L} f\big), \tag{1.7}$$

where

$$D^m f = D_{\alpha_1 \dot\alpha_1} \cdots D_{\alpha_m \dot\alpha_m} f_{\beta\gamma} + \text{symmetrized} \tag{1.8}$$

(complete symmetrization in the undotted and dotted indices, respectively), and $D_\mu = \sigma^{\alpha\dot\alpha}_\mu D_{\alpha\dot\alpha}$ is the usual Yang–Mills covariant derivative. Namely, the one-loop mixing matrix for the operators (1.7) is given by an integrable $SO(4,2) = SU(2,2)$ (non-compact!) quantum spin chain Hamiltonian with spins in the representation with Dynkin labels $[2, -3, 0]$. The anomalous dimensions are given by

$$\gamma = \frac{\alpha_s N_c}{2\pi} \left(\frac{7L}{6} - \sum_{j=1}^{M_l} \frac{2}{l_j^2 + 1} + \sum_{j=1}^{M_u} \frac{3}{u_j^2 + 9/4} \right), \tag{1.9}$$

2 There is an additional (zero-momentum) equation due to the cyclicity of the trace in the operators.

where the BAEs are now given by[2]

$$\left(\frac{l_j+i}{l_j-i}\right)^L=\prod_{\substack{k=1\\k\neq j}}^{M_l}\frac{l_j-l_k+i}{l_j-l_k-i}\prod_{k=1}^{M_u}\frac{l_j-u_k-i/2}{l_j-u_k+i/2},$$

$$\left(\frac{u_j-3i/2}{u_j+3i/2}\right)^L=\prod_{\substack{k=1\\k\neq j}}^{M_u}\frac{u_j-u_k+i}{u_j-u_k-i}\prod_{k=1}^{M_l}\frac{u_j-l_k-i/2}{u_j-l_k+i/2}\prod_{k=1}^{M_r}\frac{u_j-r_k-i/2}{u_j-r_k+i/2},$$

$$1=\prod_{\substack{k=1\\k\neq j}}^{M_r}\frac{r_j-r_k+i}{r_j-r_k-i}\prod_{k=1}^{M_u}\frac{r_j-u_k-i/2}{r_j-u_k+i/2}. \tag{1.10}$$

As noted by Beisert et al., a u-root corresponds to adding a covariant derivative $D_{1\dot{1}}$; and an l-root and an r-root flip a left-Lorentz-spin $1\to 2$ and a right-spin $\dot{1}\to\dot{2}$, respectively. The scaling dimensions and $SU(2)_L\times SU(2)_R$ quantum numbers are given by

$$D=2L+M_u,\qquad S_1=L+\frac{1}{2}M_u-M_l,\qquad S_2=\frac{1}{2}M_u-M_r, \tag{1.11}$$

respectively.

As noted in [13], the BAEs (1.10) can be obtained from those of the "beast" form of $\mathcal{N}=4$ SYM [14] by truncating the supergroup $SU(2,2|4)$ down to the Bosonic subgroup $SU(2,2)$.[3] Much attention has been focused on the S matrix of $\mathcal{N}=4$ SYM and of the corresponding string theory (see, e.g., [16]).

For the pure spin-1 problem (1.5), (1.6), the ground state for large L is described by a "sea" of approximate "2-strings" of l-roots [11,12] (in contrast to the case of the spin-1/2 antiferromagnetic XXX chain, for which the ground state is described by a sea of *real* roots). The excitations consist of "spinons" (roughly speaking, "holes" in the sea) which carry RSOS [17] quantum numbers. The spinon–spinon S matrix was found by indirect methods in [18,19], correcting the result obtained in [11] using the string hypothesis. A nonlinear integral equation (NLIE) [20,21] has been obtained for this model [22–24], which does not rely on the string hypothesis and provides a more direct way to compute the S matrix [25]. The NLIE of the $SU(2)$ sector of $\mathcal{N}=4$ SYM has been studied in [26].

For the general case (1.9), (1.10), the ground state is still a sea of approximate 2-strings of l-roots, since the u-roots contribute positively to the energy (and the r-roots do not contribute at all). Hence, there are again spinon excitations corresponding to holes in the sea. However, there are now also "magnon" excitations, corresponding to u-roots [13].

Our main objective here is to further investigate these magnon excitations, and in particular, to compute the magnon–magnon S matrix. Owing to the nontrivial nature of the ground state, this S matrix (like the spinon–spinon S matrix) must be computed with care: using the string hypothesis as in [11] gives an incorrect result. To this end, we first derive in Section 2 a set of NLIEs for the model. Although we do not invoke the string hypothesis, we do make a certain analyticity assumption in order to describe the u-roots. For simplicity, we restrict to real u-roots, and we do not consider r-roots. We then use these NLIEs to determine the energy and momentum

[3] For some early references on integrable $gl(n|m)$ spin chains, see, e.g., [15].

of the excitations (Section 3), and their S matrices (Section 4). We end in Section 5 with a brief discussion of our results.

2. Nonlinear integral equations

We restrict our attention to the case without r-roots ($M_r = 0$), for which the BAEs (1.10) reduce to

$$\left(\frac{l_j + i}{l_j - i}\right)^L = \prod_{\substack{k=1\\k\neq j}}^{M_l} \frac{l_j - l_k + i}{l_j - l_k - i} \prod_{k=1}^{M_u} \frac{l_j - u_k - i/2}{l_j - u_k + i/2}, \tag{2.1}$$

$$\left(\frac{u_j - 3i/2}{u_j + 3i/2}\right)^L = \prod_{\substack{k=1\\k\neq j}}^{M_u} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^{M_l} \frac{u_j - l_k - i/2}{u_j - l_k + i/2}. \tag{2.2}$$

We now proceed in turn to recast these two sets of BAEs in the form of NLIEs.

2.1. The first set of BAEs (2.1) and an auxiliary inhomogeneous mixed spin chain

An important hint on how to analyze the first set of BAEs (2.1) comes from rewriting it in the obviously equivalent form

$$\left(\frac{l_j + i}{l_j - i}\right)^L \prod_{k=1}^{M_u} \frac{l_j - u_k + i/2}{l_j - u_k - i/2} = \prod_{\substack{k=1\\k\neq j}}^{M_l} \frac{l_j - l_k + i}{l_j - l_k - i}. \tag{2.3}$$

We recognize these as the BAEs for an inhomogeneous "mixed" spin chain which has two types of spins: spin-1 and spin-$1/2$, with L of the former and M_u of the latter. (See, e.g., [27].) Moreover, the latter have associated "inhomogeneities" iu_k, $k = 1, \dots, M_u$.

We therefore consider an auxiliary integrable inhomogeneous mixed quantum spin chain, where the number of spin-1 and spin-$1/2$ "quantum" spaces are given respectively by L and M_u; and with spectral parameter inhomogeneities iu_k only for the spin-$1/2$ spins. This chain has two relevant transfer matrices $T_1(x)$, $T_2(x)$, corresponding to "auxiliary" spaces which are spin-$1/2$ (2-dimensional) and spin-1 (3-dimensional), respectively.

We find by standard methods that the eigenvalues of these transfer matrices (which we denote by the same notation) are given by[4]

$$T_1(x) = \psi(x - i/2)\phi(x - i)\frac{Q(x+i)}{Q(x)} + \psi(x + i/2)\phi(x + i)\frac{Q(x-i)}{Q(x)}, \tag{2.4}$$

$$\begin{aligned} T_2(x) =\ & \psi(x)\psi(x-i)\phi(x-i/2)\phi(x-3i/2)\frac{Q(x+3i/2)}{Q(x-i/2)} \\ & + \psi(x)^2\phi(x-i/2)\phi(x+i/2)\frac{Q(x+3i/2)Q(x-3i/2)}{Q(x+i/2)Q(x-i/2)} \\ & + \psi(x)\psi(x+i)\phi(x+i/2)\phi(x+3i/2)\frac{Q(x-3i/2)}{Q(x+i/2)} \\ := & \lambda_1(x) + \lambda_2(x) + \lambda_3(x), \end{aligned} \tag{2.5}$$

[4] Note that in place of the standard spectral parameter u, we introduce $u = ix$.

where

$$\phi(x)=x^L, \qquad \psi(x)=\prod_{j=1}^{M_u}(x-u_j), \qquad Q(x)=\prod_{j=1}^{M_l}(x-l_j). \tag{2.6}$$

Indeed, the BAEs obtained by demanding that $T_1(x)$ be analytic at $x=l_j$ (zeros of $Q(x)$) coincide with (2.1).

Evidently $T_2(x)$ has the common factor $\psi(x)$, which has "trivial" zeros. We therefore introduce the renormalized T_2,

$$T_2(x)=\psi(x)T_2^{(r)}(x).$$

We note that

$$S_1=S_1^z=L+\frac{M_u}{2}-M_l, \tag{2.7}$$

and we recall that the "energy" is given by (1.9).

2.1.1. Physical degrees of freedom

For simplicity, we restrict u_k to be real, and $M_u=1,2$. For now, we also assume that u_k are given by hand, with (in the case $M_u=2$) $u_1=-u_2$. We shall discuss how they should be determined later in Section 2.2.

Numerical studies for small values of L suggest that:

- For $M_u=1$, the lowest energy state in the $S_1^z=1/2$ sector is characterized by a single zero (ϑ_α) of $T_1(x)$, and a single zero (θ_h) of $T_2^{(r)}(x)$. Both of these zeros lie on the real axis.
- For $M_u=2$, the lowest energy state is in the $S_1^z=0$ sector. In the "physical strip" ($-1/2\leqslant \Im m\, x\leqslant 1/2$), $T_1(x)$ and $T_2^{(r)}(x)$ are free from zeros.
- For $M_u=2$, the second-lowest energy state is in the $S_1^z=1$ sector. It is characterized by two zeros (ϑ_α) of $T_1(x)$ and two zeros (θ_h) of $T_2^{(r)}(x)$. These zeros lie on the real axis.

These observations suggest that three sets of real parameters are needed to describe the physical degrees of freedom: $u_j, \vartheta_\alpha, \theta_h$. The first and third parameters correspond to magnon and spinon rapidities, respectively. The second parameter, which seems to correspond to excitation of the RSOS degree of freedom, is not discussed in [13].

2.1.2. The auxiliary functions and algebraic relations among them

As in previous studies [22,24], we introduce a pair of auxiliary functions

$$\mathfrak{b}_1(x):=\frac{\lambda_1(x)+\lambda_2(x)}{\lambda_3(x)} \quad \Im m\, x\geqslant 0, \qquad \bar{\mathfrak{b}}_1(x):=\frac{\lambda_2(x)+\lambda_3(x)}{\lambda_1(x)} \quad \Im m\, x\leqslant 0, \tag{2.8}$$

where $\lambda_i(x)$ are defined in (2.5). They are free from zeros and poles near the real axis. This will be apparent from the following representations,

$$\mathfrak{b}_1(x)=\frac{\phi_{-1/2}}{\psi_1\phi_{3/2}\phi_{1/2}}\frac{Q(x+3i/2)}{Q(x-3i/2)}T_1(x-i/2),$$

$$\bar{\mathfrak{b}}_1(x)=\frac{\phi_{1/2}}{\psi_{-1}\phi_{-3/2}\phi_{-1/2}}\frac{Q(x-3i/2)}{Q(x+3i/2)}T_1(x+i/2). \tag{2.9}$$

We have introduced here the abbreviated notation $\phi_a := \phi(x + ia)$, and similarly for ψ, which we shall use throughout this part of the paper.

At this stage, there seems to be no reason why the two auxiliary functions should be introduced in the corresponding half planes. This will become clear at a later stage.

The upper-case functions are also introduced: $\mathfrak{B}_1(x) = 1 + \mathfrak{b}_1(x)$, $\bar{\mathfrak{B}}_1(x) = 1 + \bar{\mathfrak{b}}_1(x)$, and the following relations are also useful:

$$T_2^{(r)}(x) = \psi_1\phi_{1/2}\phi_{3/2}\frac{Q(x - 3i/2)}{Q(x + i/2)}\mathfrak{B}_1(x) \tag{2.10}$$

$$= \psi_{-1}\phi_{-1/2}\phi_{-3/2}\frac{Q(x + 3i/2)}{Q(x - i/2)}\bar{\mathfrak{B}}_1(x). \tag{2.11}$$

Apparently $\mathfrak{B}_1(x)$ vanishes at $x = \theta_h$, but it remains nonzero at $x = u_j$.

We now define the most important functions,

$$\mathfrak{b}(x) = \mathfrak{b}_1(x + i\epsilon), \qquad \mathfrak{B}(x) = \mathfrak{B}_1(x + i\epsilon), \qquad \Im m\, x \geqslant 0, \tag{2.12}$$

$$\bar{\mathfrak{b}}(x) = \bar{\mathfrak{b}}_1(x - i\epsilon), \qquad \bar{\mathfrak{B}}(x) = \bar{\mathfrak{B}}_1(x - i\epsilon), \qquad \Im m\, x \leqslant 0. \tag{2.13}$$

Here ϵ denotes a positive quantity which is slightly larger than the deviation of the 2-strings from their "perfect" positions. Therefore $\mathfrak{B}$ would possess zeros (due to the factor $Q(x + i/2)$ in (2.10)) slightly below the real axis if it were defined in the whole complex plane. The function $\mathfrak{B}$ is, however, defined only in the upper half plane (including the real axis).

Another auxiliary function originates from the so-called fusion formula that relates the two transfer matrices,

$$T_1(x - i/2)T_1(x + i/2) = \psi_1\psi_{-1}\phi_{3/2}\phi_{-3/2} + \psi_0 T_2^{(r)}(x), \tag{2.14}$$

which can be verified using (2.4) and (2.5). For later convenience, we renormalize $T_1(x) = \prod_{\alpha=1}^{N_\vartheta} \tanh\frac{\pi}{2}(x - \vartheta_\alpha)T_1^{(r)}(x)$, and rewrite the above in the form

$$T_1^{(r)}(x - i/2)T_1^{(r)}(x + i/2) = \psi_1\psi_{-1}\phi_{3/2}\phi_{-3/2} + \psi_0 T_2^{(r)}(x)$$
$$= \psi_1\psi_{-1}\phi_{3/2}\phi_{-3/2}Y(x), \tag{2.15}$$

where we have defined the auxiliary functions

$$y(x) := \frac{\psi_0}{\psi_1\psi_{-1}\phi_{3/2}\phi_{-3/2}}T_2^{(r)}(x), \qquad Y(x) := 1 + y(x). \tag{2.16}$$

Since y possesses zeros on the real axis due to u_j and θ_h, we also define a renormalized function $y^{(r)}$

$$y(x) = \prod_{j=1}^{M_u} \tanh\frac{\pi}{2}(x - u_j) \prod_{h=1}^{N_h} \tanh\frac{\pi}{2}(x - \theta_h) y^{(r)}(x), \tag{2.17}$$

which obeys the functional relation

$$y^{(r)}(x - i/2)y^{(r)}(x + i/2) = \mathfrak{B}_1(x + i/2)\bar{\mathfrak{B}}_1(x - i/2), \tag{2.18}$$

as follows from (2.10) and (2.11).

2.1.3. Derivation of NLIE

The derivation of the NLIE can be most easily done in Fourier space. For a smooth function $f(x)$, we define

$$\hat{f}[k] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} f(x)\,dx, \qquad f(x) = \int_{-\infty}^{\infty} e^{-ikx}\hat{f}[k]\,dk. \tag{2.19}$$

We also introduce the special notation

$$\widehat{dlf}[k] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\big[\ln f(x)\big]'\,dx, \tag{2.20}$$

which will be frequently used below.

It is convenient to introduce "shifted" Q functions,

$$q_1(x) := Q(x - i/2 - i\epsilon), \qquad q_2(x) := Q(x + i/2 + i\epsilon). \tag{2.21}$$

By definition, q_1 is Analytic and NonZero (ANZ) for $\Im m\, x \leqslant 0$, while q_2 is ANZ for $\Im m\, x \geqslant 0$. We therefore have by Cauchy's theorem the important property

$$\widehat{dlq_2}[k > 0] = \widehat{dlq_1}[k < 0] = 0. \tag{2.22}$$

Similarly,

$$\begin{aligned} &\widehat{dl\psi_a}[k > 0] = \widehat{dl\phi_a}[k > 0] = 0 \quad \text{for } a > 0, \\ &\widehat{dl\psi_a}[k < 0] = \widehat{dl\phi_a}[k < 0] = 0 \quad \text{for } a < 0. \end{aligned} \tag{2.23}$$

We slightly shift the arguments in (2.10), (2.11)

$$T_2^{(r)}(x + i\epsilon) = \psi_{1+\epsilon}\phi_{1/2+\epsilon}\phi_{3/2+\epsilon}\frac{q_1(x - i + 2i\epsilon)}{q_2(x)}\mathfrak{B}(x), \tag{2.24}$$

$$T_2^{(r)}(x - i\epsilon) = \psi_{-1-\epsilon}\phi_{-1/2-\epsilon}\phi_{-3/2-\epsilon}\frac{q_2(x + i - 2i\epsilon)}{q_1(x)}\bar{\mathfrak{B}}(x). \tag{2.25}$$

We then use the result

$$\frac{1}{2\pi}\int_{C_\epsilon} e^{ikx}\big[\ln T_2^{(r)}(x)\big]'\,dx = i\sum_h e^{ik\theta_h}, \tag{2.26}$$

where we choose the contour C_ϵ as in Fig. 1, and we obtain the following

$$\begin{aligned} \widehat{dlq_1}[k > 0] = {} & \frac{\widehat{dl\psi}_{-1-\epsilon} + \widehat{dl\phi}_{-3/2-\epsilon} + \widehat{dl\phi}_{-1/2-\epsilon}}{1 + e^{-k}} + \frac{\widehat{dl\bar{\mathfrak{B}}}[k] - e^{-2k\epsilon}\widehat{dl\mathfrak{B}}[k]}{1 + e^{-k}} \\ & - i\sum_h \frac{e^{ik\theta_h - k\epsilon}}{1 + e^{-k}}, \end{aligned} \tag{2.27}$$

$$\begin{aligned} \widehat{dlq_2}[k < 0] = {} & \frac{\widehat{dl\psi}_{1+\epsilon} + \widehat{dl\phi}_{3/2+\epsilon} + \widehat{dl\phi}_{1/2+\epsilon}}{1 + e^{k}} + \frac{\widehat{dl\mathfrak{B}}[k] - e^{2k\epsilon}\widehat{dl\bar{\mathfrak{B}}}[k]}{1 + e^{k}} \\ & + i\sum_h \frac{e^{ik\theta_h + k\epsilon}}{1 + e^{k}}. \end{aligned} \tag{2.28}$$

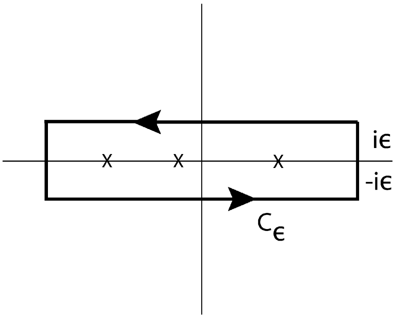

Fig. 1. Integration contour.

In addition, from (2.15), one derives

$$\widehat{dlT}_1[k] = \frac{\widehat{dl\psi}_{\mp 1} + \widehat{dl\phi}_{\mp 3/2}}{e^{k/2} + e^{-k/2}} + \frac{\widehat{dlY}}{e^{k/2} + e^{-k/2}} + i\sum_\alpha \frac{e^{ik\vartheta_\alpha + k/2}}{e^{k/2} + e^{-k/2}}, \tag{2.29}$$

where $-(k > 0)$ and $+(k < 0)$.

We shift the arguments in (2.9)

$$\mathfrak{b}(x) = \frac{\phi_{-1/2+\epsilon}}{\psi_{1+\epsilon}\phi_{3/2+\epsilon}\phi_{1/2+\epsilon}} \frac{q_2(x+i)}{q_1(x-i+2i\epsilon)} T_1(x - i/2 + i\epsilon),$$
$$\bar{\mathfrak{b}}(x) = \frac{\phi_{1/2-\epsilon}}{\psi_{-1-\epsilon}\phi_{-3/2-\epsilon}\phi_{-1/2-\epsilon}} \frac{q_1(x-i)}{q_2(x+i-2i\epsilon)} T_1(x + i/2 - i\epsilon), \tag{2.30}$$

and then take the Fourier transformation. The substitution of (2.27), (2.28) and (2.29) into the resultant transformation then leads to the NLIE in Fourier space,

$$\widehat{dlb}[k > 0] = \frac{\widehat{dl\phi}_{-1/2+\epsilon}}{1 + e^{-k}} + i\sum_h \frac{e^{ik\theta_h + \epsilon k}}{1 + e^k} + i\sum_\alpha \frac{e^{ik\vartheta_\alpha + \epsilon k}}{e^{k/2} + e^{-k/2}}$$
$$+ \frac{e^{-k/2+\epsilon k}}{e^{k/2} + e^{-k/2}}\widehat{dlY}[k] + \frac{1}{e^k + 1}\big(\widehat{dl\mathfrak{B}}[k] - e^{2k\epsilon}\widehat{dl\bar{\mathfrak{B}}}[k]\big), \tag{2.31}$$

$$\widehat{dlb}[k < 0] = -\frac{\widehat{dl\phi}_{1/2+\epsilon}}{1 + e^{k}} + i\sum_h \frac{e^{ik\theta_h + \epsilon k}}{1 + e^{-k}} + i\sum_\alpha \frac{e^{ik\vartheta_\alpha + \epsilon k}}{e^{k/2} + e^{-k/2}}$$
$$+ \frac{e^{-k/2+\epsilon k}}{e^{k/2} + e^{-k/2}}\widehat{dlY}[k] + \frac{1}{e^{-k} + 1}\big(\widehat{dl\mathfrak{B}}[k] - e^{2k\epsilon}\widehat{dl\bar{\mathfrak{B}}}[k]\big). \tag{2.32}$$

Interestingly, although a contribution from the inhomogeneities (ψ) appeared during the calculation, it canceled in the final form. An equation for y is immediately derived from (2.18),

$$\widehat{dly}[k] = i\sum_h \frac{e^{ik\theta_h}}{1 + e^{-k}} + i\sum_j \frac{e^{iku_j}}{1 + e^{-k}}$$
$$+ \frac{e^{k(1/2-\epsilon)}}{e^{k/2} + e^{-k/2}}\widehat{dl\mathfrak{B}}[k] + \frac{e^{-k(1/2-\epsilon)}}{e^{k/2} + e^{-k/2}}\widehat{dl\bar{\mathfrak{B}}}[k]. \tag{2.33}$$

In the original coordinate space, the resultant equations read

$$\ln \mathfrak{b}(x) = i D_b(x+i\epsilon) + \int_{-\infty}^{\infty} G_s(x-x') \ln \mathfrak{B}(x')\, dx' - \int_{-\infty}^{\infty} G_s(x-x'+2i\epsilon) \ln \bar{\mathfrak{B}}(x')\, dx'$$
$$+ \int_{-\infty}^{\infty} K(x-x'-i/2+i\epsilon) \ln Y(x')\, dx', \tag{2.34}$$

$$\ln y(x) = i D_y(x) + \int_{-\infty}^{\infty} K(x-x'+i/2-i\epsilon) \ln \mathfrak{B}(x')\, dx'$$
$$+ \int_{-\infty}^{\infty} K(x-x'-i/2+i\epsilon) \ln \bar{\mathfrak{B}}(x')\, dx', \tag{2.35}$$

where

$$G_s(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1+e^{|k|}} dk,$$
$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\cosh(k/2)} e^{-ikx}\, dk = \frac{1}{2\cosh(\pi x)}. \tag{2.36}$$

The source term in (2.34) consists of the bulk ("driving") contribution and the contribution from the hole excitations,

$$D_b(x) = D^{(b)}_{\text{bulk}}(x) + D^{(b)}_{\text{hole}}(x) - \frac{\pi}{2} N_h, \tag{2.37}$$

where

$$D^{(b)}_{\text{bulk}}(x) = L\chi_K(x), \qquad D^{(b)}_{\text{hole}}(x) = \sum_{\alpha=1}^{N_\vartheta} \chi_K(x-\vartheta_\alpha) + \sum_{h=1}^{N_h} \chi(x-\theta_h), \tag{2.38}$$

and

$$\chi'_K(x) = 2\pi K(x), \qquad \chi'(x) = 2\pi G_s(x). \tag{2.39}$$

In particular, on suitable domains (containing the positive real axis),

$$\chi_K(x) = \frac{1}{i} \ln \tanh\bigl(\pi(x-i/2)/2\bigr) = i \ln \frac{\sinh(\pi(x+i/2)/2)}{\sinh(\pi(x-i/2)/2)} + \frac{\pi}{2}$$
$$= \arctan\bigl(\sinh(\pi x)\bigr) - \frac{\pi}{2}, \tag{2.40}$$

where $\chi_K(0) \equiv -\pi/2$, and also $\chi(0) \equiv 0$. The source term in (2.35) is given by

$$D_y(x) = \sum_{h=1}^{N_h} \chi_K(x-\theta_h+i/2) + \sum_{j=1}^{M_u} \chi_K(x-u_j+i/2). \tag{2.41}$$

The parameters $(u_j, \vartheta_\alpha, \theta_h)$ must actually be determined again by NLIEs. Indeed, (2.10) implies that the hole rapidities θ_h are determined by

$$\mathfrak{b}(\theta_h - i\epsilon) = \mathfrak{b}_1(\theta_h) = -1, \tag{2.42}$$

which also leads to the determination of the spinon–spinon and spinon–magnon scattering matrices, as discussed in Section 4.

In order to fix the parameters ϑ_α, we need another NLIE. We consider the most natural auxiliary function $\mathfrak{a}(x)$, defined by[5]

$$\mathfrak{a}(x) := \frac{\lambda_1(x+i/2)}{\lambda_2(x+i/2)} = \frac{\lambda_2(x-i/2)}{\lambda_3(x-i/2)}, \tag{2.43}$$

where again $\lambda_i(x)$ are defined in (2.5). From (2.4) we have

$$T_1(x) = \psi_{1/2}\phi_1 \frac{Q(x-i)}{Q(x)}[1+\mathfrak{a}(x)]. \tag{2.44}$$

Hence, the zeros of T_1 on the real axis ϑ_α satisfy

$$\mathfrak{a}(\vartheta_\alpha) = -1. \tag{2.45}$$

We omit the derivation of the NLIE for $\mathfrak{a}(x)$ for $|\Im m\, x| < 1/2$, which is similar to the one for the trigonometric and homogeneous case considered in [24]. The result is

$$\ln \mathfrak{a}(x) = i D_a(x) + \int_{-\infty}^{\infty} K(x-x'-i\epsilon) \ln \mathfrak{B}(x')\, dx' - \int_{-\infty}^{\infty} K(x-x'+i\epsilon) \ln \bar{\mathfrak{B}}(x')\, dx', \tag{2.46}$$

where the source term is given by

$$D_a(x) = \sum_{h=1}^{N_h} \chi_K(x-\theta_h) + \sum_{j=1}^{M_u} \chi_K(x-u_j). \tag{2.47}$$

2.2. *The second set of BAEs (2.2)*

We finally consider an equation to fix the magnon rapidities u_j. For this purpose, we propose an expression for the transfer matrix eigenvalues similar to the one for the $su(3)$ spin chain,[6]

$$\begin{aligned}\tau(x) &= \phi(x-i)\frac{Q(x+i)}{Q(x)} + \phi(x+i)\frac{\psi(x+i/2)}{\psi(x-i/2)}\frac{Q(x-i)}{Q(x)} + \phi(x-2i)\frac{\psi(x-3i/2)}{\psi(x-i/2)} \\ &:= \tau_1(x) + \tau_2(x) + \tau_3(x).\end{aligned} \tag{2.48}$$

Indeed, demanding analyticity of $\tau(x)$ at $x = l_j$ (zeros of $Q(x)$) gives the BAEs (2.1), while demanding analyticity at $x = u_j + i/2$ (zeros of $\psi(x-i/2)$) gives the BAEs (2.2).

Because of its similarity to the $su(3)$ transfer matrix eigenvalue, we shall assume that $\tau(x)$ is ANZ in the strip $-1/2 \leqslant \Im m\, x \leqslant 1/2$, which is indeed the analyticity property for the $su(3)$ case. This assumption can in principle be checked numerically for small values of L. However,

[5] As discussed further in Section 2.3, one can verify numerically that $\frac{1}{i}\ln \mathfrak{a}(x)$ and also $\Re e[\frac{1}{i}\ln \mathfrak{b}(x)]$ are increasing functions of x.

[6] We expect that, starting from a suitable $su(2,1)$ R matrix, a transfer matrix can be constructed with eigenvalues (2.48). However, we have not attempted to carry out this construction.

we have so far not succeeded to do so, due to the difficulty of finding numerical solutions of the BAEs (2.1), (2.2).

This assumption leads to a simple determination of u_j as follows. Let us consider an auxiliary function introduced in studies of the supersymmetric tJ model [28] and the $su(3)$ vertex model [29],

$$\mathfrak{c}(x) := \frac{\tau_3(x+i/2)}{\tau_1(x+i/2)+\tau_2(x+i/2)}. \tag{2.49}$$

It is easy to check that this can be rewritten in terms of $T_1(x)$ in (2.4),

$$\mathfrak{c}(x) = \frac{\psi(x-i)\phi(x-3i/2)}{T_1(x+i/2)}. \tag{2.50}$$

We then have

$$\mathfrak{c}(u_j) = -1, \tag{2.51}$$

which follows from

$$\mathfrak{C}(x) = 1+\mathfrak{c}(x) = \frac{\tau(x+i/2)}{\tau_1(x+i/2)+\tau_2(x+i/2)} = \frac{\tau(x+i/2)\psi(x)}{T_1(x+i/2)}. \tag{2.52}$$

From our above assumption on the analyticity of $\tau(x)$, the zeros of $\mathfrak{C}(x)$ near the real axis are determined by those of $\psi(x)$, namely u_j.

The NLIE for $\mathfrak{c}$ is obtained from the knowledge of T_1. The result is

$$\ln\mathfrak{c}(x) = iD_c(x) - \int_{-\infty}^{\infty} K(x-x'+i/2)\ln Y(x')\,dx', \tag{2.53}$$

where the source term is given by

$$D_c(x) = L\chi_2(x) + \sum_{j=1}^{M_u}\chi_{3/2}(x-u_j) + \sum_{\alpha=1}^{N_\vartheta}\chi_K(x-\vartheta_\alpha) - \frac{\pi}{2}(L+M_u), \tag{2.54}$$

and

$$\chi_a'(x) = 2\pi K_a(x), \quad K_a(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-a|k|-ikx}}{2\cosh\frac{k}{2}}dk,$$
$$\chi_a(0) \equiv 0, \quad a = 3/2, 2. \tag{2.55}$$

2.3. *Counting functions and counting equations*

So-called counting equations relating the various types of Bethe roots and excitations in a given state can be derived from corresponding counting functions associated with the auxiliary functions. These counting equations help determine the spins of the excitations.

We continue to restrict to the case of real u-roots and no r-roots. As in previous studies [21,24,25], it is convenient to classify l-roots according to their imaginary parts as follows:

2-*strings*: pairs of complex-conjugate roots $x_j \pm iy_j$ with $0 < y_j - 1/2 \ll 1$, $j = 1,\ldots,M_2/2$,
real roots: $\Im m\, l_j = 0$, $j = 1,\ldots,M_{\mathrm{real}}$,

inner roots: $|\Im m\, l_j| < 1/2$, $j = 1, \dots, M_I$,
close roots: $1/2 < |\Im m\, l_j| < 3/2$, $j = 1, \dots, M_C$,
wide roots: $|\Im m\, l_j| > 3/2$, $j = 1, \dots, M_W$.

Hence,

$$M_l = M_{\text{real}} + M_2 + M_I + M_C + M_W. \tag{2.56}$$

It is also convenient to introduce the functions

$$\theta_{\mp}(x, \alpha) = \frac{1}{i} \ln\left(\mp \frac{x - i\alpha}{x + i\alpha}\right). \tag{2.57}$$

Note that $\theta_-(x, \alpha) = 2 \arctan(x/\alpha)$ has branch points in the complex x plane at $x = \pm i\alpha$; following [21], we choose the corresponding branch cuts to be parallel to the real axis, extending from $i\alpha$ to $+\infty + i\alpha$, and from $-\infty - i\alpha$ to $-i\alpha$. This function has a discontinuity of -2π when crossing the cuts from below. Similarly, we add to $\theta_+(x, \alpha)$ a 2π-discontinuity at $x = 0$ so that it is a continuous function of x.

We define the counting function $z_{\mathfrak{a}}(x)$ associated with the auxiliary function $\mathfrak{a}(x)$ (2.43) by

$$z_{\mathfrak{a}}(x) = \frac{1}{i} \operatorname{Log} \mathfrak{a}(x) = L\theta_-(x, 1) - \sum_{j=1}^{M_l} \theta_-(x - l_j, 1) + \sum_{j=1}^{M_u} \theta_+(x - u_j, 1/2). \tag{2.58}$$

We have verified numerically for various states that $z_{\mathfrak{a}}(x)$ is a continuous increasing function of x. This function "counts" zeros of $T_1(x)$ and real l-roots. That is,

$$z_{\mathfrak{a}}(x_j) = 2\pi I_j^{\mathfrak{a}}, \tag{2.59}$$

where $I_j^{\mathfrak{a}}$ is integer ($S_1 - S_2$ odd) or half-odd integer ($S_1 - S_2$ even) if x_j is a zero of $T_1(x)$ or a real l-root. Defining integers or half-odd integers $I_{\max}^{\mathfrak{a}}$ and $I_{\min}^{\mathfrak{a}}$ by

$$z_{\mathfrak{a}}(+\infty) = 2\pi\left(I_{\max}^{\mathfrak{a}} + \frac{1}{2}\right), \qquad z_{\mathfrak{a}}(-\infty) = 2\pi\left(I_{\min}^{\mathfrak{a}} - \frac{1}{2}\right), \tag{2.60}$$

it follows from (2.58) and (2.59), respectively, that

$$I_{\max}^{\mathfrak{a}} - I_{\min}^{\mathfrak{a}} + 1 = S_1 + S_2 + M_b = N_\vartheta + M_{\text{real}}, \tag{2.61}$$

where M_b is the number of l-roots l_j with $|\Im m\, l_j| > 1$. We therefore arrive at the first counting equation

$$N_\vartheta = S_1 + S_2 + M_b - M_{\text{real}}. \tag{2.62}$$

Similarly, we define the counting function $z_{\mathfrak{b}}(x)$ associated with the auxiliary function $\mathfrak{b}_1(x)$ (2.8) by

$$\begin{aligned} z_{\mathfrak{b}}(x) &= \Re e \frac{1}{i} \operatorname{Log} \mathfrak{b}_1(x) \\ &= \Re e \left\{ \frac{1}{i} \ln\left[1 + \frac{1}{\mathfrak{a}(x - i/2)}\right] + \sum_{j=1}^{M_u} \theta_+(x - u_j, 1) + L\left[\theta_-(x, 1/2) + \theta_-(x, 3/2)\right] \right. \\ &\qquad \left. - \sum_{j=1}^{M_l} \left[\theta_-(x - l_j, 1/2) + \theta_-(x - l_j, 3/2)\right] \right\}. \end{aligned} \tag{2.63}$$

The presence of the first term generally requires the introduction of further discontinuities. We have verified numerically that $z_{\mathfrak{b}}(x)$ is also a continuous increasing function of x. This function "counts" zeros of $T_2^{(r)}(x)$ and centers of 2-strings and inner pairs. Proceeding as before, we find

$$\begin{aligned} I^{\mathfrak{b}}_{\max} - I^{\mathfrak{b}}_{\min} + 1 &= 2S_1 + M_C + 2M_W + (M_2 + M_I)/2 \\ &= N_h + (M_2 + M_I)/2. \end{aligned} \tag{2.64}$$

We therefore arrive at the second counting equation

$$N_h = 2S_1 + M_C + 2M_W. \tag{2.65}$$

Finally, we define the counting function $z_{\mathfrak{c}}(x)$ associated with the auxiliary function $\mathfrak{c}(x)$ (2.49) by

$$\begin{aligned} z_{\mathfrak{c}}(x) &= \Re e \frac{1}{i} \operatorname{Log} \mathfrak{c}(x) \\ &= \Re e \left\{ -\frac{1}{i} \ln\bigl[1 + \mathfrak{a}(x + i/2)\bigr] + \sum_{j=1}^{M_u} \theta_+(x - u_j, 1) + L\theta_-(x, 3/2) \right. \\ &\quad \left. - \sum_{j=1}^{M_l} \theta_-(x - l_j, 1/2) \right\}. \end{aligned} \tag{2.66}$$

We have verified numerically (using for the first term the same discontinuities introduced for the first term in (2.63)) that $z_{\mathfrak{c}}(x)$ is a continuous increasing function of x. Assuming

$$I^{\mathfrak{c}}_{\max} - I^{\mathfrak{c}}_{\min} + 1 = L + M_u - M_{\text{real}} - (M_2 + M_I)/2, \tag{2.67}$$

which can also be verified numerically, we recover the result

$$M_u = 2S_2. \tag{2.68}$$

3. Spin, energy and momentum of excitations

We now compute the excitations' spin, energy and momentum, which enter into the computation of the S matrix. Our results agree (except for some minor discrepancies) with those obtained previously using the string hypothesis.

We can infer the spins of the excitations with the help of the counting equations found in Section 2.3. The second counting equation (2.65) implies that a spinon has $S_1 = 1/2$. Indeed, $N_h = 1$ requires $S_1 = 1/2$ (and $M_C = M_W = 0$); $N_h = 2$ requires either $S_1 = 0$ or $S_1 = 1$, etc. Note that all the terms on the RHS of (2.65) are nonnegative. Evidently, a spinon also has $S_2 = 0$. The fact that a spinon has spin-$1/2$ was found using the string hypothesis by Takhtajan [11].

Similarly, the third counting equation (2.68) implies that a magnon has $S_2 = 1/2$, and evidently $S_1 = 0$. This result was found using the string hypothesis by Beisert et al. [13].

The spin of the ϑ particle is not determined by the first counting equation (2.65), since not all the terms on the RHS are nonnegative. Nevertheless, an analysis of various examples suggests that this particle has $S_1 = S_2 = 0$.

By the definition in [13], the energy (E) is related to the anomalous dimension (1.9) by $\gamma = \frac{\alpha_s N_c}{2\pi} E$, and is therefore given by[7]

$$E = -\sum_{j=1}^{M_l} \frac{2}{l_j^2 + 1} + \sum_{j=1}^{M_u} \frac{3}{u_j^2 + 9/4}. \tag{3.1}$$

We can relate this to the derivate of the eigenvalue $T_2(x)$ (2.5) at $x = i/2$,

$$E = i \frac{d}{dx} \ln T_2(x)|_{x=i/2} - \frac{3L}{2} + \sum_{j=1}^{M_u} \left[\frac{3}{u_j^2 + 9/4} + i \left(\frac{1}{u_j - i/2} + \frac{1}{u_j - 3i/2} \right) \right]. \tag{3.2}$$

Recalling the definition of the auxiliary function $y(x)$ (2.16), we see that

$$E = i \frac{d}{dx} \ln y(x)|_{x=i/2} - 2L + \sum_{j=1}^{M_u} \left(\frac{3}{u_j^2 + 9/4} - \frac{1}{u_j^2 + 1/4} \right). \tag{3.3}$$

We observe from (2.20) that

$$\frac{d}{dx} \ln y(x) = \int_{-\infty}^{\infty} dk e^{-ikx} \widehat{dly}[k], \tag{3.4}$$

and substitute our result for $\widehat{dly}[k]$ (2.33) to obtain

$$E = -2L + \sum_{j=1}^{M_u} \left[\frac{\pi}{\cosh(\pi u_j)} + \frac{3}{u_j^2 + 9/4} - \frac{1}{u_j^2 + 1/4} \right] + \sum_h \frac{\pi}{\cosh(\pi \theta_h)} + \cdots, \tag{3.5}$$

where the ellipsis ($\cdots$) represents the Casimir energy contribution. We conclude that the energy of a spinon is

$$\varepsilon_h(\theta) = \frac{\pi}{\cosh(\pi\theta)}, \tag{3.6}$$

and the energy of a magnon is

$$\varepsilon_u(u) = \frac{\pi}{\cosh(\pi u)} + \frac{3}{u^2 + 9/4} - \frac{1}{u^2 + 1/4}, \tag{3.7}$$

in agreement with Eqs. (6.15), (6.32) in Beisert et al. [13], respectively, up to a factor 2. The spinon result (3.6) was first found by Takhtajan [11]. We remark that

$$\varepsilon_u(u) = 2\pi K_2(u), \tag{3.8}$$

where $K_2(u)$ is the kernel introduced in (2.55). Evidently there is no ϑ-dependent contribution in (3.5), which implies that the ϑ "particle" does not carry energy.

The momentum is given by[8]

$$P = \frac{1}{i} \left[\sum_{j=1}^{M_l} \ln\left(\frac{l_j + i}{l_j - i} \right) + \sum_{j=1}^{M_u} \ln\left(\frac{u_j - 3i/2}{u_j + 3i/2} \right) \right] \pmod{2\pi}. \tag{3.9}$$

---

[7] For convenience, we drop the constant term $7L/6$ in the expression for E. This definition of energy is (for the l-roots) a factor 2 larger than the one in [11].

[8] This definition of momentum differs (for the l-roots) by an overall sign from the one in [11].

We can evaluate it in similar fashion. Indeed, we find that

$$P = \frac{1}{i}\ln y(i/2) + \frac{1}{i}\sum_{j=1}^{M_u}\big[\ln e_{-3}(u_j) + \ln e_1(u_j)\big] + L\pi, \tag{3.10}$$

where we have introduced the notation

$$e_n(u) = \frac{u + in/2}{u - in/2}. \tag{3.11}$$

Proceeding as before, we arrive at the result

$$P = L\pi + \sum_{j=1}^{M_u}\big[\chi_K(u_j) + q_3(u_j) - q_1(u_j)\big] + \sum_h \chi_K(\theta_h) + \cdots, \tag{3.12}$$

where $\chi_K(x)$ is defined in (2.39), and $q_n(x)$ is defined by

$$q_n(x) = \pi + i\ln e_n(x) \quad n > 0, \qquad q_{-n}(x) = -q_n(x), \qquad q_0(x) = 0. \tag{3.13}$$

It is an odd function of x, and satisfies

$$q_n(x) = 2\arctan(2x/n), \quad n \neq 0. \tag{3.14}$$

We conclude that the momentum of a spinon is

$$p_h(\theta) = \chi_K(\theta), \tag{3.15}$$

and the momentum of a magnon is

$$p_u(u) = \chi_K(u) + q_3(u) - q_1(u), \tag{3.16}$$

in agreement with Eqs. (6.15), (6.32) in [13], respectively, up to an overall sign. Corresponding to the energy result (3.8), we observe that

$$p_u(u) = \chi_2(u), \tag{3.17}$$

where $\chi_2(u)$ is defined in (2.55). The ϑ particle also does not carry momentum.

4. *S* matrix

We finally turn to the problem of computing the scattering amplitudes for the various excitations.

4.1. Spinon–spinon

It is convenient to review the computation of the spinon–spinon S matrix [18,19] using the NLIE approach [25]. Let $\theta_{h_1}, \theta_{h_2}$ denote the rapidities of the two spinons. Since $\mathfrak{b}(\theta_{h_1} - i\epsilon) = -1$ (2.42), the $\ln \mathfrak{b}$ equation (2.34) implies

$$i\pi = iD_b(\theta_{h_1}) + \int_{-\infty}^{\infty} dx'\, K(\theta_{h_1} - x' - i/2)\ln Y(x'), \tag{4.1}$$

since the convolution terms involving $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ become exponentially small in the IR limit. Neglecting the convolution term in the $\ln y$ equation (2.35), one obtains

$$y(x) = \tanh\big(\pi(x-\theta_{h_1})/2\big)\tanh\big(\pi(x-\theta_{h_2})/2\big), \tag{4.2}$$

and therefore

$$Y(x) = 1 + y(x) = \frac{\cosh(\pi(x-(\theta_{h_1}+\theta_{h_2})/2))}{\cosh(\pi(x-\theta_{h_1})/2)\cosh(\pi(x-\theta_{h_2})/2)}. \tag{4.3}$$

We now exponentiate both sides of (4.1), and note using (2.37), (2.38) that

$$D_b(\theta_{h_1}) = L\chi_K(\theta_{h_1}) + \chi(\theta) - \pi, \quad \theta = \theta_{h_1} - \theta_{h_2}. \tag{4.4}$$

With the help of the momentum expression (3.15), we compare the result with the Yang equation

$$e^{iLp_h(\theta_{h_1})} S_{h,h}(\theta) = 1. \tag{4.5}$$

We conclude that the S matrix is given (up to a constant) by

$$S_{h,h}(\theta) = e^{i\chi(\theta)} S_{\mathrm{RSOS}}(\theta), \tag{4.6}$$

where

$$S_{\mathrm{RSOS}}(\theta) = e^{\int_{-\infty}^{\infty} dx' K(\theta_{h_1}-x'-i/2)\ln Y(x')} = e^{-\frac{i}{2}[\psi_0(\theta)-\varphi_2(\theta)]} = e^{-i[\psi_0(\theta)-\varphi_4(\theta)]}, \tag{4.7}$$

and

$$\psi_0(x) = \arctan\sinh(\pi x/2) = i\ln\frac{\sinh(\pi(i+x)/4)}{\sinh(\pi(i-x)/4)}, \tag{4.8}$$

$$\varphi_n(x) = \int_0^\infty dk\,\frac{\sin(kx)\sinh((n-1)k/2)}{k\sinh(nk/2)\cosh(k/2)}, \tag{4.9}$$

with $\varphi_4(x) = (\varphi_2(x)+\psi_0(x))/2$. The convolution integrals are performed using the results collected in the appendix. The result (4.7) is (up to a crossing factor, and a rescaling of the rapidity by π) one of the kink-kink scattering amplitudes of the tricritical Ising model perturbed by the operator $\Phi_{(1,3)}$ [17], which appears also in the soliton–soliton S matrix of the supersymmetric sine-Gordon model [18]. We note that

$$\chi(\theta) = \frac{1}{i}\ln\frac{\Gamma(1+i\theta/2)\Gamma(1/2-i\theta/2)}{\Gamma(1-i\theta/2)\Gamma(1/2+i\theta/2)}, \tag{4.10}$$

which is (up to the same rescaling of the rapidity by π) the soliton–soliton scattering phase of the sine-Gordon model [30] in the isotropic limit $\beta^2 \to 8\pi$.

If we also consider the $\ln y$ equation with an additional $i\pi$ term, then the RHS of (4.2) acquires a minus sign. The corresponding amplitudes can be computed along similar lines [25]. However, for simplicity, we restrict our attention to the $\ln y$ equation without this additional $i\pi$ term.

4.2. Spinon–magnon

Let θ_{h_1}, u_1 denote the rapidities of the spinon and magnon, respectively. The spinon–magnon S matrix can be computed in two different ways. One way is to start from $\mathfrak{b}(\theta_{h_1} - i\epsilon) = -1$, which again leads to (4.1). The $\ln y$ equation implies

$$y(x) = \tanh\bigl(\pi(x-\theta_{h_1})/2\bigr)\tanh\bigl(\pi(x-u_1)/2\bigr), \tag{4.11}$$

and therefore

$$Y(x) = 1 + y(x) = \frac{\cosh(\pi(x-(\theta_{h_1}+u_1)/2))}{\cosh(\pi(x-\theta_{h_1})/2)\cosh(\pi(x-u_1)/2)}. \tag{4.12}$$

Moreover, now $D_b(\theta_{h_1}) = L\chi_K(\theta_{h_1}) = Lp_h(\theta_{h_1})$, up to an additive constant. Proceeding as before, we obtain the result

$$S_{h,u}(\theta) = S_{\mathrm{RSOS}}(\theta), \tag{4.13}$$

where now $\theta = \theta_{h_1} - u_1$, and $S_{\mathrm{RSOS}}(\theta)$ is given by (4.7). That is, in contrast to the spinon–spinon S matrix (4.6), the spinon–magnon S matrix consists only of the RSOS factor.

A second way to compute the spinon–magnon S matrix is to start from $\mathfrak{c}(u_1) = -1$ (2.51), which together with the $\ln\mathfrak{c}$ equation (2.53) imply

$$i\pi = iD_c(u_1) + \int_{-\infty}^{\infty} dx'\,K(u_1 - x' - i/2)\ln Y(x'). \tag{4.14}$$

We exponentiate both sides of this equation, and note that

$$D_c(u_1) = L\chi_2(u_1) = Lp_u(u_1), \tag{4.15}$$

where we have made use of (2.54) and the momentum result (3.17). Comparing with the corresponding Yang equation, we recover the same result, i.e.,

$$S_{u,h}(\theta) = S_{\mathrm{RSOS}}(\theta), \tag{4.16}$$

where now $\theta = u_1 - \theta_{h_1}$.

4.3. Magnon–magnon

Let u_1, u_2 be the rapidities of the two magnons. The $\ln y$ equation (2.35) implies

$$y(x) = \tanh\bigl(\pi(x-u_1)/2\bigr)\tanh\bigl(\pi(x-u_2)/2\bigr), \tag{4.17}$$

and

$$Y(x) = 1 + y(x) = \frac{\cosh(\pi(x-(u_1+u_2)/2))}{\cosh(\pi(x-u_1)/2)\cosh(\pi(x-u_2)/2)}. \tag{4.18}$$

The condition $\mathfrak{c}(u_1) = -1$ (2.51) and the $\ln\mathfrak{c}$ equation (2.53) again give (4.14), where now (cf. (4.15))

$$D_c(u_1) = Lp_u(u_1) + \chi_{3/2}(\theta), \tag{4.19}$$

with $\theta = u_1 - u_2$. Proceeding as before, we conclude that the magnon–magnon S matrix is given by

$$S_{u,u}(\theta) = e^{i\chi_{3/2}(\theta)} S_{\mathrm{RSOS}}(\theta), \tag{4.20}$$

where $S_{\mathrm{RSOS}}(\theta)$ is given by (4.7). We note that

$$\chi_{3/2}(\theta) = \frac{1}{i} \ln \frac{\Gamma(-1/2 + i\theta/2)\Gamma(-i\theta/2)}{\Gamma(-1/2 - i\theta/2)\Gamma(i\theta/2)} + \pi, \tag{4.21}$$

and that $s(\theta) \equiv e^{i\chi_{3/2}(\theta)}$ has the crossing property

$$s(i - \theta) = \left(\frac{1 - i\theta}{2 + i\theta}\right) s(\theta). \tag{4.22}$$

Hence, $s(\theta)/(1 + i\theta/2)$ is crossing invariant.

We have considered so far the composite operators containing only $D_{\alpha\dot{1}}$ covariant derivatives and computed the S matrix amplitude between them. In principle, one would need to add r-roots to compute amplitudes for the derivatives carrying the right-spin state $\dot{2}$. But this can be done, without adding r-roots, by using the $SU(2)_R$ symmetry. The "vertex" part of the S matrix is in fact a 4×4 matrix which can be fixed completely by the $SU(2)_R$ symmetry along with factorizability (i.e., Yang–Baxter equation), unitarity and crossing,

$$\frac{s(\theta)}{1 + i\theta/2} (\mathcal{P} + i\theta/2), \tag{4.23}$$

where $\mathcal{P}$ is the permutation matrix.

4.4. ϑ-spinon and ϑ-magnon

The condition $\mathfrak{a}(\vartheta_\alpha) = -1$ (2.45) together with the $\ln \mathfrak{a}$ equation (2.46) imply that the S matrices $S_{\vartheta,h}$ and $S_{\vartheta,u}$ are identical, and are given by

$$S(\theta) = \frac{\sinh(\pi(\theta/2 - i/4))}{\sinh(\pi(\theta/2 + i/4))}. \tag{4.24}$$

The same result can also be obtained starting from (2.42), (2.34) (for $S_{h,\vartheta}$) and from (2.51), (2.53) (for $S_{u,\vartheta}$). Since there is no ϑ-dependent contribution in the source term of the $\ln \mathfrak{a}$ equation (2.46), there is no nontrivial ϑ–ϑ scattering.

5. Discussion

We have proposed a set of NLIEs (2.34)–(2.41), (2.46), (2.47), (2.53)–(2.55) to describe the QCD spin chain of Beisert et al. [13]. We have used these NLIEs to compute S matrix elements for excitations of this model, as shown in detail in Section 4. The consistency of our results ($S_{a,b} = S_{b,a}$ for particles a and b of different types) provides further support for the validity of these NLIEs.

Many questions remain to be addressed. It should be possible to generalize this work along the lines [31] and compute the boundary S matrix for the open QCD spin chain corresponding to operators with quarks at the ends. The magnon–magnon S matrix (4.20), (4.21) has an infinite number of singularities (starting at $\theta = \pm 2i$), which can presumably be interpreted as magnon–magnon bound states ("breathers"). The energy and momentum of these breathers was computed using the string hypothesis in [13]. It would be interesting to analyze these excitations without invoking the string hypothesis, and to determine their S matrices. It would also be interesting to consider the effects of higher loops ([7] and [13] worked only to leading order in the 't Hoof coupling) and to better understand the significance of these results for QCD, as well as for the full $\mathcal{N} = 4$ SYM theory and for the corresponding string theory.

Acknowledgements

One of us (C.A.) thanks Shizuoka University and the University of Miami for support. This work was supported in part by a Korea Research Foundation Grant funded by the Korean government (MOEHRD) (KRF-2006-312-C00096) (C.A.), by the National Science Foundation under Grants PHY-0244261 and PHY-0554821 (R.N.), and by the Ministry of Education of Japan, a Grant-in-Aid for Scientific Research 17540354 (J.S.).

Appendix A. Convolutions

The convolution integrals involving the kernel $K(x-\frac{i}{2})=\frac{i}{2\sinh\pi x}$ can be evaluated using the following results

$$\frac{i}{2}\int_{-\infty}^{\infty} dx' \frac{\ln\cosh(\pi(x'-i\epsilon))}{\sinh(\pi(x-x'+i\epsilon))} = -\frac{i}{2}\arctan\sinh(\pi x)+\frac{1}{2}\ln\cosh(\pi x), \tag{A.1}$$

$$\frac{i}{2}\int_{-\infty}^{\infty} dx' \frac{\ln\cosh(\pi(x'-i\epsilon)/2)}{\sinh(\pi(x-x'+i\epsilon))} = -\frac{i}{2}\varphi_2(x)+\frac{1}{2}\ln\cosh(\pi x/2), \tag{A.2}$$

$$\frac{i}{2}\int_{-\infty}^{\infty} dx' \frac{\ln\sinh(\pi(x'-i\epsilon))}{\sinh(\pi(x-x'+i\epsilon))} = \frac{1}{2}\ln\sinh(\pi x)-\frac{1}{2}\ln\tanh(\pi x/2)-\frac{i\pi}{4}, \tag{A.3}$$

$$\frac{i}{2}\int_{-\infty}^{\infty} dx' \frac{\ln\sinh(\pi(x'-i\epsilon)/2)}{\sinh(\pi(x-x'+i\epsilon))} = \frac{i}{2}\varphi_2(x)+\frac{1}{2}\ln\cosh(\pi x/2)-\frac{i\pi}{4}, \tag{A.4}$$

where ϵ is a small positive number, and $\varphi_2(x)$ is given by (4.9).

References

[1] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. B 72 (1974) 461.
[2] A.M. Polyakov, String representations and hidden symmetries for gauge fields, Phys. Lett. B 82 (1979) 247.
[3] Y.M. Makeenko, A.A. Migdal, Exact equation for the loop average in multicolor QCD, Phys. Lett. B 88 (1979) 135.
[4] L.N. Lipatov, High energy asymptotics of multi-colour QCD and exactly solvable lattice models, JETP Lett. 59 (1994) 596, hep-th/9311037.
[5] L.D. Faddeev, G.P. Korchemsky, High-energy QCD as a completely integrable model, Phys. Lett. B 342 (1995) 311, hep-th/9404173.
[6] A.V. Belitsky, Renormalization of twist-three operators and integrable lattice models, Nucl. Phys. B 574 (2000) 407, hep-ph/9907420;
S.E. Derkachov, G.P. Korchemsky, A.N. Manashov, Evolution equations for quark–gluon distributions in multi-color QCD and open spin chains, Nucl. Phys. B 566 (2000) 203, hep-ph/9909539;
A.V. Belitsky, V.M. Braun, A.S. Gorsky, G.P. Korchemsky, Integrability in QCD and beyond, Int. J. Mod. Phys. A 19 (2004) 4715, hep-th/0407232.
[7] G. Ferretti, R. Heise, K. Zarembo, New integrable structures in large-N QCD, Phys. Rev. D 70 (2004) 074024, hep-th/0404187.
[8] J.A. Minahan, K. Zarembo, The Bethe ansatz for $N=4$ super-Yang–Mills, JHEP 0303 (2003) 013, hep-th/0212208;
J.A. Minahan, A brief introduction to the Bethe ansatz in $\mathcal{N}=4$ super-Yang–Mills, J. Phys. A 39 (2006) 12657.
[9] A.B. Zamolodchikov, V.A. Fateev, Model factorized S matrix and an integrable Heisenberg chain with spin 1, Sov. J. Nucl. Phys. 32 (1980) 298.

[10] P.P. Kulish, N.Yu. Reshetikhin, E.K. Sklyanin, Yang–Baxter equation and representation theory. I, Lett. Math. Phys. 5 (1981) 393;
P.P. Kulish, E.K. Sklyanin, Quantum Spectral Transform Method, Recent Developments, Lecture Notes in Physics, vol. 151, Springer, 1982, p. 61.
[11] L.A. Takhtajan, The picture of low-lying excitations in the isotropic Heisenberg chain with arbitrary spins, Phys. Lett. A 87 (1982) 479.
[12] H.M. Babujian, Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spins S, Phys. Lett. A 90 (1982) 479;
H.M. Babujian, Exact solution of the isotropic Heisenberg chain with arbitrary spins: Thermodynamics of the model, Nucl. Phys. B 215 (1983) 317.
[13] N. Beisert, G. Ferretti, R. Heise, K. Zarembo, One-loop QCD spin chain and its spectrum, Nucl. Phys. B 717 (2005) 137, hep-th/0412029.
[14] N. Beisert, M. Staudacher, The $\mathcal{N}=4$ SYM integrable super spin chain, Nucl. Phys. B 670 (2003) 439, hep-th/0307042.
[15] G.V. Uimin, One-dimensional problem for $S=1$ with modified antiferromagnetic Hamiltonian, JETP Lett. 12 (1970) 225;
C.K. Lai, Lattice gas with nearest-neighbor interaction in one dimension with arbitrary statistics, J. Math. Phys. 15 (1974) 1675;
B. Sutherland, Model for a multicomponent quantum system, Phys. Rev. B 12 (1975) 3795;
P.P. Kulish, E.K. Sklyanin, Solutions of the Yang–Baxter equation, J. Sov. Math. 19 (1982) 1596;
P.P. Kulish, Integrable graded magnets, J. Sov. Math. 35 (1986) 2648;
F.H.L. Essler, V.E. Korepin, Higher conservation laws and algebraic Bethe Ansätze for the supersymmetric t–J model, Phys. Rev. B 46 (1992) 9147, hep-th/9207007;
A. Foerster, M. Karowski, Algebraic properties of the Bethe ansatz for an spl(2,1)-supersymmetric t–J model, Nucl. Phys. B 396 (1993) 611;
P.B. Ramos, M.J. Martins, One parameter family of an integrable spl(2|1) vertex model: Algebraic Bethe ansatz and ground state structure, Nucl. Phys. B 474 (1996) 678, hep-th/9604072;
M.P. Pfannmüller, H. Frahm, Algebraic Bethe ansatz for gl(2,1) invariant 36-vertex models, Nucl. Phys. B 479 (1996) 575, cond-mat/9604082;
H. Saleur, The continuum limit of sl(N/K) integrable super spin chains, Nucl. Phys. B 578 (2000) 552, solv-int/9905007.
[16] M. Staudacher, The factorized S-matrix of CFT/AdS, JHEP 0505 (2005) 054, hep-th/0412188;
N. Beisert, The $SU(2|2)$ dynamic S-matrix, hep-th/0511082;
R.A. Janik, The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry, Phys. Rev. D 73 (2006) 086006, hep-th/0603038;
N. Beisert, B. Eden, M. Staudacher, Transcendentality and crossing, J. Stat. Mech. 0701 (2007) P021, hep-th/0610251.
[17] G.E. Andrews, R.J. Baxter, P.J. Forrester, Eight vertex SOS model and generalized Rogers–Ramanujan type identities, J. Stat. Phys. 35 (1984) 193;
A.B. Zamolodchikov, Fractional-spin integrals of motion in perturbed conformal field theory, in: H. Guo, Z. Qiu, H. Tye (Eds.), Fields, Strings and Quantum Gravity, Gordon and Breach, 1989.
[18] C. Ahn, D. Bernard, A. LeClair, Fractional supersymmetries in perturbed coset CFTs and integrable soliton theory, Nucl. Phys. B 346 (1990) 409;
C. Ahn, Complete S-matrices of supersymmetric sine-Gordon theory and perturbed superconformal minimal model, Nucl. Phys. B 354 (1991) 57.
[19] N. Reshetikhin, S-matrices in integrable models of isotropic magnetic chains. I, J. Phys. A 24 (1991) 3299.
[20] A. Klümper, M.T. Batchelor, An analytic treatment of finite-size corrections in the spin-1 antiferromagnetic XXZ chain, J. Phys. A 23 (1990) L189;
A. Klümper, M.T. Batchelor, P.A. Pearce, Central charges of the 6- and 19-vertex models with twisted boundary conditions, J. Phys. A 24 (1991) 3111;
C. Destri, H. de Vega, New thermodynamic Bethe ansatz equations without strings, Phys. Rev. Lett. 69 (1992) 2313, hep-th/9203064.
[21] D. Fioravanti, A. Mariottini, E. Quattrini, F. Ravanini, Excited state Destri–De Vega equation for sine-Gordon and restricted sine-Gordon models, Phys. Lett. B 390 (1997) 243, hep-th/9608091;
C. Destri, H. de Vega, Non linear integral equation and excited-states scaling functions in the sine-Gordon model, Nucl. Phys. B 504 (1997) 621, hep-th/9701107;

G. Feverati, Finite volume spectrum of sine-Gordon model and its restrictions, hep-th/0001172.
[22] J. Suzuki, Spinons in magnetic chains of arbitrary spins at finite temperatures, J. Phys. A 32 (1999) 2341.
[23] C. Dunning, Finite size effects and the supersymmetric sine-Gordon models, J. Phys. A 36 (2003) 5463, hep-th/0210225.
[24] J. Suzuki, Excited states nonlinear integral equations for an integrable anisotropic spin 1 chain, J. Phys. A 37 (2004) 11957, hep-th/0410243.
[25] Á. Hegedűs, F. Ravanini, J. Suzuki, Exact finite size spectrum in super sine-Gordon model, Nucl. Phys. B 763 (2007) 330, hep-th/0610012.
[26] G. Feverati, D. Fioravanti, P. Grinza, M. Rossi, On the finite size corrections of anti-ferromagnetic anomalous dimensions in $\mathcal{N} = 4$ SYM, JHEP 0605 (2006) 068, hep-th/0602189;
G. Feverati, D. Fioravanti, P. Grinza, M. Rossi, Hubbard's adventures in $\mathcal{N} = 4$ SYM-land? Some non-perturbative considerations on finite length operators, J. Stat. Mech. 0702 (2007) P001, hep-th/0611186.
[27] H.J. de Vega, F. Woynarovich, New integrable quantum chains combining different kinds of spins, J. Phys. A 25 (1992) 4499.
[28] G. Jüttner, A. Klümper, J. Suzuki, Exact thermodynamics and Luttinger liquid properties of the integrable t–J model, Nucl. Phys. B 487 (1997) 650, cond-mat/9611058.
[29] A. Fujii, A. Klümper, Anti-symmetrically fused model and non-linear integral equations in the three-state Uimin–Sutherland model, Nucl. Phys. B 546 (1999) 751, cond-mat/9811234.
[30] A.B. Zamolodchikov, Al.B. Zamolodchikov, Factorized S matrices in two-dimensions as the exact solutions of certain relativistic quantum field models, Ann. Phys. 120 (1979) 253.
[31] C. Ahn, R.I. Nepomechie, J. Suzuki, Finite size effects in the spin-1 XXZ and supersymmetric sine-Gordon models with Dirichlet boundary conditions, Nucl. Phys. B 767 (2007) 250, hep-th/0611136.