Multi-matrix model and 2D Toda multi-component hierarchy

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We study the integrability of the hermitian matrix-chain model at finite N. The integrable system, constructed from the matrix integrals using orthogonal polynomials is identified with the two-dimensional Toda system with multi-component hierarchy. We derive the Lax equations, the zero curvature conditions and an infinite number of conserved quantities for this 2D Toda hierarchy. The partition function of the matrix model is proved to be the "tau-function" of this Toda system. Also, using our formalism, we derive the Virasoro constraints on the partition function of the multi-matrix model for the first time.

1. Recent progress on matrix models shows the intimate relationship of the model with classical integrable systems. From the original non-perturbative formulation [1,2] of two-dimensional (2D) pure quantum gravity using the matrix models, it is noticed that in the double scaling limit the exact solvability is closely related to an integrable system; the string equations for specific heat are expressed with the conserved charges of the KdV equations. The Rutgers group [3] further noticed in the continuum limit that the change of the potential is governed by the renormalization group (RG) equations, which are identified again with the KdV hierarchy [4].

These important results on the continuum one-matrix model have been generalized in two directions: one is to extend the one-matrix model to the multi-matrix model in the continuum limit to describe 2D quantum gravity coupled with minimal ($c<1$) matter and the other approach is to understand the integrability of the matrix models at finite N, the size of the matrix. The former approach was initiated in a nice paper by Douglas [5], who showed that the continuum limit of these models can be described in terms of the Heisenberg operators. He also exhibited some interesting connections with the KdV hierarchy. The connections of the matrix models with the KdV hierarchy have been studied from a different point of view; from the Schwinger–Dyson equations of the one-matrix integration the Virasoro constraints on the partition function have been derived. Furthermore, one of these constraints is shown to be the KdV equation [5]. For the multi-matrix model, it is suggested that in addition to the Virasoro constraints there are W_n algebra constraints on the partition function without any derivation.

The study on the matrix models at finite N is motivated by the wish to understand better the origin of the integrable systems that appear at the level of the matrix integrals. This approach can answer many interesting questions which are not clear in the effective continuum theory. Recent progress in this direction is due to three independent papers, by the ITEP group, by Martinec, and by Alvarez-Gaumé et al. [7–9]. For the one-matrix model, these authors have found that the underlying integrable system is the one-dimensional Toda hierarchy and the partition function is identified with the "τ-function" of the Toda hierarchy. The Virasoro constraints are also derived at finite N, which are consistent with the continuum results [7,10,11]. The explicit continuum limit is taken for the one-matrix model with even potential which reduces to the Volterra hierarchy at finite N to derive the string equation and the KdV flow equations [9].

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For the multi-matrix model, there are few rigorous results as is the case for the continuum multi-matrix model. Partial results on the two-matrix model are as follows: the underlying differential equation is identified with the 2D Toda equation [7] and the Lax equations and zero curvature conditions for the 2D Toda hierarchy are derived [8]. But the Virasoro constraints conjectured in the continuum limit and the general integrable systems underlying the multi-matrix model are totally missing.

In this letter we follow the approach of refs. [7–9] to extend the formalism to the hermitian matrix-chain model to fill the gap in the understanding of the multi-matrix model both at finite N and in the continuum limit. We identify that the underlying integrable system is the 2D Toda multi-component hierarchy [12], for which the Lax equations and the zero curvature conditions are expressed with several copies of time variables. The one-component hierarchy is the usual 2D Toda hierarchy associated with the two-matrix model considered in ref. [8]. The partition function is identified with the "τ-function" of this 2D Toda hierarchy in the same way as the one- and the two-matrix model. Furthermore, we show that the string equations at finite N are compatible with the Toda flow equations, generalizing the results of the one-matrix model. Using our formalism, we derive the Virasoro constraints on the partition function. Also, we will briefly mention the consistency of the results at finite N with those in the continuum limit.

2. The hermitian matrix-chain model is defined by the following partition function:

$$Z_N^{(p)} = \int \prod_{a=1}^{p} \mathrm{d}M_a \exp\left[\mathrm{Tr}\left(\sum_{a=1}^{p-1} c_a M_a M_{a+1} - \sum_{a=1}^{p} V_a(M_a)\right)\right], \quad V_a(M_a) = \sum_{k=1}^{\infty} t_k^{(a)} (M_a)^k . \tag{1}$$

We introduce the enlarged parameter space spanned by $\{t^{(a)}\}$, $\{c_a\}$, where the theory is determined by the RG flows with respect to these parameters. After angle integrations, we can express the partition function as an integration of diagonal elements,

$$Z_N^{(p)} = \int \prod_{i=1}^{N} \prod_{a=1}^{p} \mathrm{d}x_a^{(i)}\, \Delta(x_p)\Delta(x_1) \exp\left[\sum_{i=1}^{N}\left(\sum_{a=1}^{p-1} c_a x_a^{(i)} x_{a+1}^{(i)} - \sum_{a=1}^{p} V_a(x_a^{(i)})\right)\right],$$

$$\Delta(x_a) = \prod_{i>j} (x_a^{(i)} - x_a^{(j)}) . \tag{2}$$

Using the standard technique for the multi-matrix model [13], we introduce the orthogonal polynomials, defined as follows:

$$\exp(\phi_m)\,\delta_{mn} = \int \prod_{a=1}^{p} \mathrm{d}x_a \exp\left(\sum_{a=1}^{p-1} c_a x_a x_{a+1} - \sum_{a=1}^{p} V_a(x_a)\right) \bar{P}_m(x_p) P_n(x_1) \equiv \langle m | n \rangle . \tag{3}$$

In terms of these eigenvalues ϕ_n, the partition function is expressed by

$$Z_N^{(p)}[\{t^{(a)}\}, \{c_a\}] = \mathrm{const} \cdot \prod_{n=0}^{N-1} \exp(\phi_n) . \tag{4}$$

We introduce the matrices $Q^{(a)}$, $P^{(a)}$:

$$x_{a+1} \mathcal{M}^{(a)}[P_n](x_{a+1}) = \sum_l \mathcal{M}^{(a)}[P_l](x_{a+1}) Q^{(a+1)}{}_{ln} ,$$

$$\frac{\partial}{\partial x_{a+1}} \mathcal{M}^{(a)}[P_n](x_{a+1}) = \sum_l \mathcal{M}^{(a)}[P_l](x_{a+1}) P^{(a+1)}{}_{ln} , \tag{5}$$

where the involution of the polynomials is defined by the recursive relation

$$\mathcal{M}^{(a)}[P_n](x_{a+1}) = \int \mathrm{d}x_a \exp[-U_a(x_a, x_{a+1})]\, \mathcal{M}^{(a-1)}[P_n](x_a) , \tag{6}$$

$$U_a(x_a, x_{a+1}) = V_a(x_a) - c_a x_a x_{a+1} , \tag{6 cont'd}$$

with $\mathcal{M}^{(0)}[P_n] = P_n$. These two Heisenberg operators are related to each other by

$$P^{(a)} = c_{a-1} Q^{(a-1)}, \quad [P^{(a)}, Q^{(a)}] = 1 , \tag{7}$$

and the $Q^{(a)}$'s are satisfying the following recursion formula:

$$c_a Q^{(a+1)} = V'_a(Q^{(a)}) - c_{a-1} Q^{(a-1)} . \tag{8}$$

Therefore, the string equations $[P^{(a)}, Q^{(a)}] = 1$ are recursively related to each other. Due to relation (7), we will consider the Heisenberg operators $Q^{(a)}$ only. We will show that the operators $Q^{(a)}$ satisfy the Lax equations and the zero curvature conditions. Using this, we can generate sets of an infinite number of conserved charges. By deriving the 2D Toda equation as a special case of the Lax equations, we will connect the multi-matrix model with the 2D Toda system with multi-component hierarchy.

First, we consider the RG flows of the $Q^{(a)}$'s. We prove that the Lax equations are given by

$$\frac{\partial}{\partial t_l^{(b)}} Q^{(a)} - [Q^{(b)l}{}_>, Q^{(a)}] \quad \text{if } b \geqslant a, \quad \frac{\partial}{\partial t_l^{(b)}} Q^{(a)} = [Q^{(b)l}{}_\leqslant, Q^{(a)}] \quad \text{if } b \leqslant a . \tag{9}$$

Throughout this paper, we use the convention that $M = M_> + M_\leqslant$ where $>$, $<$, and $=$ mean the upper triangular, lower triangular, and diagonal elements, respectively. (The matrix $M_\leqslant$ means lower triangular and diagonal part of M.) We can generalize eq. (9) to the powers of $Q^{(a)}$ as follows:

$$\frac{\partial}{\partial t_l^{(b)}} Q^{(a)k} = -[Q^{(b)l}{}_>, Q^{(a)k}] \quad \text{if } b \geqslant a, \quad \frac{\partial}{\partial t_l^{(b)}} Q^{(a)k} = [Q^{(b)l}{}_\leqslant, Q^{(a)k}] \quad \text{if } b \leqslant a . \tag{10}$$

These Lax equations define the 2D Toda system with multi-component hierarchy defined in ref. [12]. There are p copies of time variables corresponding with index a.

To prove these Lax equations, we use eq. (5) successively to derive

$$x_{a+1}^k \mathcal{M}^{(a)}[P_n](x_{a+1}) = \sum_l \mathcal{M}^{(a)}[P_l](x_{a+1}) Q^{(a+1)k}{}_{ln} . \tag{11}$$

Taking a $t_l^{(b)}$ derivative on eq. (11), one can find

$$\frac{\partial}{\partial t_l^{(b)}} Q^{(a)k}{}_{mn} = \theta_{ab}[Q^{(b)l}, Q^{(a)k}]_{mn} - \sum_{j=1}^{\infty} f_{m+j,l}^{(j,b)} Q^{(a)k}{}_{m+j,n} + \sum_{j=1}^{n} f_{n,l}^{(j,b)} Q^{(a)k}{}_{m,n-j} , \tag{12}$$

with $\theta_{ab} = 1$ if $a \geqslant b$, 0 if $a < b$. The coefficients $f_n^{(j)}$ are necessary to express the derivatives on the polynomials P_n, which are defined by

$$\frac{\partial P_n(x)}{\partial t_l^{(b)}} = \sum_{j=1}^{n} f_{n,l}^{(j,b)} P_{n-j}(x) .$$

To find expressions for the $f_{n,l}^{(j,b)}$'s, we differentiate eq. (3) with respect to $t_k^{(a)}$ to get

$$\frac{\partial \phi_n}{\partial t_k^{(a)}} = -Q^{(a)k}{}_{nn}, \quad f_{n,k}^{(j,a)} = Q^{(a)k}{}_{n-j,n} . \tag{13}$$

Combining eqs. (12) and (13), one can obtain the Lax equations (9) and (10).

The next thing we want to show is the zero curvature conditions for the consistency of the Lax flow equations (9) and (10):

$$\left[\frac{\partial}{\partial t_k^{(a)}} + Q_>^{(a)}, \frac{\partial}{\partial t_k^{(b)}} + Q_>^{(b)}\right] = 0, \quad \left[\frac{\partial}{\partial t_k^{(a)}} - Q_\leqslant^{(a)}, \frac{\partial}{\partial t_k^{(b)}} - Q_\leqslant^{(b)}\right] = 0 , \tag{14}$$

$$\left[\frac{\partial}{\partial t_k^{(a)}}+Q_>^{(a)}, \frac{\partial}{\partial t_k^{(b)}}-Q_\leqslant^{(b)}\right]=0 . \tag{14 cont'd}$$

We considered the case $a \geqslant b$ without losing any generality. Note that $[\partial/\partial t_k^{(a)}-Q_\leqslant^{(a)}, \partial/\partial t_k^{(b)}+Q_>^{(b)}]=0$ need not be satisfied for the consistency of the Lax equations. The zero curvature conditions, eq. (14), can be rewritten using the Lax equations as follows:

$$-[Q^{(a)k}{}_>, Q^{(b)l}]_> - [Q^{(b)l}{}_\leqslant, Q^{(a)k}]_> + [Q^{(a)k}{}_>, Q^{(b)l}{}_>]=0 ,$$
$$[Q^{(a)k}{}_>, Q^{(b)l}]_\leqslant + [Q^{(b)l}{}_\leqslant, Q^{(a)k}]_\leqslant + [Q^{(a)k}{}_\leqslant, Q^{(b)l}{}_\leqslant]=0 ,$$
$$[Q^{(a)k}{}_>, Q^{(b)l}]_\leqslant - [Q^{(b)l}{}_\leqslant, Q^{(a)k}]_> - [Q^{(a)k}{}_>, Q^{(b)l}{}_\leqslant]=0 . \tag{15}$$

Using that for any two matrices A, B: $[A_> B_>]_\leqslant=0$, $[A_\leqslant B_\leqslant]_>=0$, it is straightforward to prove eq. (15). This shows that the Lax equations are consistent with the zero curvature conditions for the 2D Toda multi-component hierarchy.

We can rewrite eq. (10) as the hierarchy equations for ϕ_n using eq. (13) to get

$$\frac{\partial^2 \phi_n}{\partial t_k^{(a)}\, \partial t_l^{(b)}}=[Q^{(b)l}{}_>, Q^{(a)k}{}_<]_{nn} \quad \text{if } b \geqslant a, \qquad \frac{\partial^2 \phi_n}{\partial t_k^{(a)}\, \partial t_l^{(b)}}=-[Q^{(b)l}{}_<, Q^{(a)k}{}_>]_{nn} \quad \text{if } b \leqslant a . \tag{16}$$

Especially, for $a=1$, $b=p$ and $k=l=1$, the equation becomes

$$\frac{\partial^2 \phi_n}{\partial t_1^{(1)}\, \partial t_1^{(p)}}=\sum_{m>n} Q^{(p)}{}_{nm} Q^{(1)}{}_{mn} - \sum_{m<n} Q^{(1)}{}_{nm} Q^{(p)}{}_{mn} . \tag{17}$$

Using the recursion formula for $P_n(x_1)$

$$x_1 P_n(x_1)=\sum_{l=0}^{n+1} P_l(x_1) Q^{(1)}{}_{ln}=P_{n+1}(x_1)+\dots , \tag{18}$$

and similarly for $\bar{P}_m(x_p)$, one can easily find

$$Q^{(1)}{}_{mn}=\delta_{m,n+1}, \quad Q^{(p)}{}_{nm}=\delta_{m,n+1} \exp(\phi_m-\phi_n) \quad (m>n) . \tag{19}$$

Substituting this into eq. (17), we derive the 2D Toda lattice [SU (∞) Toda] equation

$$\frac{\partial^2 \phi_n}{\partial t_1^{(1)}\, \partial t_1^{(p)}}=\exp(\phi_{n+1}-\phi_n)-\exp(\phi_n-\phi_{n-1}) . \tag{20}$$

As considered in ref. [9] for the one-matrix model with even potential ($t_{\text{odd}}^{(a)}=0$), the Toda equation is reduced to the two-step Toda equation

$$\frac{\partial^2 \phi_n}{\partial t_2^{(1)}\, \partial t_2^{(p)}}=\exp(\phi_{n+2}-\phi_n)-\exp(\phi_n-\phi_{n-2}) . \tag{21}$$

3. In addition to $t_k^{(a)}$, the multi-matrix model contains other time variables, c_a. The flow in the c_a direction can be derived in the same way as above. Taking a c_a derivative on eqs. (3) and (11), we obtain the following Lax equations:

$$\frac{\partial}{\partial c_b} Q^{(a)k}=[(Q^{(b+1)}Q^{(b)})_>, Q^{(a)k}] \qquad \text{if } b \geqslant a ,$$
$$=-[(Q^{(b+1)}Q^{(b)})_\leqslant, Q^{(a)k}] \quad \text{if } b<a . \tag{22}$$

Taking t_a and c_a derivatives on the recursion relation

$$x_{a+1}\int \mathrm{d}x_a \exp(-U_a)\, x_a \mathcal{M}^{(a-1)}[P_n](x_a) = \sum_l \mathcal{M}^{(a)}[P_l](x_{a+1})\,[Q^{(a+1)}Q^{(a)}]_{ln}\,, \tag{23}$$

we derive the additional flows:

$$\begin{aligned}
\frac{\partial}{\partial t_l^{(b)}}(Q^{(a+1)}Q^{(a)}) &= -[Q^{(b)l}{}_{>}, Q^{(a+1)}Q^{(a)}] && \text{if } b>a\,,\\
&= [Q^{(b)l}{}_{\leqslant}, Q^{(a+1)}Q^{(a)}] && \text{if } b\leqslant a\,,\\
\frac{\partial}{\partial c_b}(Q^{(a+1)}Q^{(a)}) &= [Q^{(b+1)}Q^{(b)}{}_{>}, Q^{(a+1)}Q^{(a)}] && \text{if } b\geqslant a\,,\\
&= -[Q^{(b+1)}Q^{(b)}{}_{\leqslant}, Q^{(a+1)}Q^{(a)}] && \text{if } b\leqslant a\,.
\end{aligned} \tag{24}$$

Using these Lax flows, one can prove general zero curvature conditions in the same manner as before, which can be expressed as follows:

$$\begin{aligned}
&\frac{\partial}{\partial \xi_k^{(a)}} M_l^{(b)} - \frac{\partial}{\partial \xi_l^{(b)}} M_k^{(a)} - [M_k^{(a)}, M_l^{(b)}] = 0\,,\\
&M_k^{(a)} \equiv -Q^{(a)k}{}_{>}, Q^{(a)k}{}_{\leqslant} \quad \text{for } \xi_k^{(a)} \equiv t_k^{(a)}\,,\\
&M_k^{(a)} \equiv [Q^{(a+1)}Q^{(a)}]_{>}, -[Q^{(a+1)}Q^{(a)}]_{\leqslant} \quad \text{for } \xi_k^{(a)} \equiv c_a\,.
\end{aligned} \tag{25}$$

The integrability of the 2D Toda multi-component hierarchy can be characterized by the infinite number of conserved charges, which can be constructed directly from the Lax equations (10), (22), and (24):

$$\mathcal{Q}_k^{(a)} = \mathrm{Tr}[Q^{(a)k}], \quad \mathrm{Tr}[Q^{(a+1)}Q^{(a)}], \quad k=1,2,\ldots, \quad \frac{\partial \mathcal{Q}_k^{(a)}}{\partial t_l^{(b)}} = \frac{\partial \mathcal{Q}_k^{(a)}}{\partial c_b} = 0\,. \tag{26}$$

Using these conserved charges, one can define the "τ-function" of the 2D Toda multi-component hierarchy as follows:

$$\tau[\{t^{(a)}\},\{c_a\}] = \langle \Psi|\exp\left(\mathrm{Tr}\left[-\sum_{a=1}^{p}\sum_k t_k^{(a)}Q^{(a)k} + \sum_{a=1}^{p-1} c_a Q^{(a+1)}Q^{(a)}\right]\right)|\Psi\rangle\,. \tag{27}$$

This is a different definition of τ-function from the one appearing in the literature about Toda hierarchies [8]. One can prove that this "τ-function" is the same as the partition function $Z_N^{(p)}$ of the multi-matrix model, if we identify $\Delta(X^{(1)}) = \langle X^{(1)}|\Psi\rangle$ and $\Delta(X^{(p)}) = \langle \Psi|X^{(p)}\rangle$ using eq. (2). We will show later that the partition function, or "τ-function", satisfies the Virasoro constraints, which are consistent with the Toda hierarchy equations (16).

All the previous arguments on the RG flows of the $Q^{(a)}$'s can equally hold for the conjugate operators $P^{(a)}$ because, as one can see in eq. (5), the flows generated by the $t_k^{(a)}$'s only depend on the properties of orthogonal polynomials and associated potentials. [This is more obvious in eq. (7).] We can prove the compatibility of the string equations and the Toda flows for the multi-matrix models as was done for the one-matrix model in ref. [8]. Taking the $t_l^{(b)}$ (or c_b) derivative of the string equations $[P^{(a)}, Q^{(a)}]=1$ for $b\geqslant a$,

$$\begin{aligned}
&\left[\frac{\partial P^{(a)}}{\partial t_l^{(b)}}, Q^{(a)}\right] + \left[P^{(a)}, \frac{\partial Q^{(a)}}{\partial t_l^{(b)}}\right] = -[[Q^{(b)l}{}_{>}, P^{(a)}], Q^{(a)}] - [P^{(a)}, [Q^{(b)l}{}_{>}, Q^{(a)}]]\\
&\quad = -[[Q^{(a)}, P^{(a)}], Q^{(b)l}{}_{>}] = 0\,,
\end{aligned} \tag{28}$$

where we used the RG flow equations in the first line and the Jacobi identity in the second. All other cases follow in a similar way. In the continuum limit, this compatibility of the string equation with the KdV flows was shown in ref. [14]. This means that the solutions to the string equation are not unique; these are parametrized by an infinite set of deformations, given by the KdV flows [14].

As we have shown in eqs. (10), (22) and (24), there are two hierarchy directions: one generated by the time variables $t_k^{(a)}$ and the other by the c_a's. The integrable system associated with the multi-matrix model is more general than this, in that the multi-component hierarchy includes only the former hierarchy direction [12]. Still, we refer to this system as the 2D Toda multi-component hierarchy because the general structure does not change. (One may consider c_a as one of the $t_k^{(a)}$'s, say, $t_0^{(a)}$.)

The flows in the c_a direction have been noticed by Douglas in the continuum limit [5], which he claimed to be the KdV hierarchy:

$$\frac{\partial Q}{\partial c_a} = [\mathcal{O}_a, Q], \quad \frac{\partial P}{\partial c_a} = [\mathcal{O}_a, P] , \tag{29}$$

where the $\mathcal{O}_a$'s are given by the differential operators $Q^{a/q}{}_+$ in the continuum limit. Our result in eq. (22) identifies this operator to be $(Q^{(a+1)}Q^{(a)})_>$ at finite N.

4. Using our formalism, we can derive the Virasoro constraints on the partition function of the multi-matrix model. Although some attempts have been made for this case [7,10], our result is the first rigorous derivation. We report only the results here, deferring a detailed derivation elsewhere. Let us consider the following change of integration variables in eq. (2):

$$x_a^{(i)} \to x_a^{(i)} + \epsilon_a [x_a^{(i)}]^{k+1}, \quad k \geqslant -1 . \tag{30}$$

By setting the coefficients of $\mathcal{O}(\epsilon_a)$ to zero, we can derive the following constraints on the partition function τ:

$$\mathcal{L}_k^{(a)}\tau = 0, \quad k \geqslant -1 ,$$

$$\mathcal{L}_k^{(a)} = -(k+1)[1-\tfrac{1}{2}(\delta_{a,1}+\delta_{a,p})]\frac{\partial}{\partial t_k^{(a)}} + \tfrac{1}{2}(\delta_{a,1}+\delta_{a,p}) \sum_{\substack{i,j=0\\ i+j=k}}^{k} \frac{\partial^2}{\partial t_i^{(a)}\,\partial t_j^{(a)}} + \sum_l l t_l^{(a)} \frac{\partial}{\partial t_{k+l}^{(a)}}$$

$$+ \sum_{n=0}^{N-1} [c_a Q^{(a+1)} Q^{(a)k+1} + c_{a-1} Q^{(a)k+1} Q^{(a-1)}]_{nn} . \tag{31}$$

Note that the last term in eq. (31) is not a differential operator but just an algebraic function of the $t_k^{(a)}$'s, which can be in principle determined by the recursion relation of the $Q^{(a)}$'s. In fact the $Q^{(a)}$'s are complicated functions of the $t_k^{(a)}$'s and the c_a's.

Amusingly, we need not know the $Q^{(a)}$'s explicitly to derive the Virasoro relations between the $\mathcal{L}_k^{(a)}$'s. The only necessary relations are the Lax equations for the $Q^{(a)}$'s. After a lengthy and non-trivial computation, we can prove

$$[\mathcal{L}_n^{(a)}, \mathcal{L}_m^{(b)}] = \delta_{ab}(n-m)\mathcal{L}_{n+m}^{(a)} . \tag{32}$$

These are a direct sum of the p copies of the classical Virasoro algebras. The continuum limit of this Virasoro algebra can be derived straightforwardly. We do not find the W_n algebra structure conjectured in ref. [6] in the continuum limit. Another change of variables like eq. (30) might lead to this, which is not clear at this moment.

To compare with some known results on the multi-matrix model in the continuum limit [5,14], we need to take the double scaling limit as was done for the one-matrix model in ref. [9]. This requires to understand the 2D Toda hierarchy in the Poisson bracket formalism. This result on the multi-matrix model in the continuum limit would be useful in order to understand non-perturbative aspects of 2D gravity coupled with matter. In particular, if we consider another limit $p \to \infty$, we can handle a more realistic model describing the $c=1$ matter coupled to 2D gravity. The underlying integrable lattice models associated with this theory will be very important. We hope to report these results on the continuum limit in a later publication [15].

As we showed, the multi-matrix model is connected with a new kind of classical integrable system; the ordinary 2D Toda hierarchy is extended to the generalized one, the multi-component hierarchy. The matrix model approach may be a powerful method to deal with this integrable system. We think this is true for general classical integrable systems. For example, it can be useful to understand how the solutions of non-linear differential equations are related; in the matrix models, if we add an additional potential $V'(x)$ to the existing potential $\sum_k t_k^{(a)} x^k$, none of the above derivations is changed i.e., they yield the same 2D Toda hierarchies, while the explicit solution ϕ_n depends on the extra potential $V'(x)$. This hints at some applicability of the matrix models to understand classical integrable systems.

In summary, in this letter we analyzed the integrability of the hermitian matrix-chain model, considering their RG flows. We constructed the Lax equations and the zero curvature conditions. By deriving the Toda equation in this way, we connect the integrability of the matrix model with the 2D Toda multi-component hierarchy. As usual, the partition function is related to the τ-function of the 2D Toda system. Also, we derived the underlying Virasoro structure to confirm some conjectures made earlier.

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